WIGNER TIME DELAY DISTRIBUTION FOR ONE-DIMENSIONAL RANDOM POTENTIALS

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The distribution of the time delay for one-dimensional random potentials is universal in the high energy or weak disorder limit. The analytical results which are here reviewed are in excellent agreement with extensive numerical simulations. We also provide a physical argument which explains in a quantitative way the origin of the exponential divergence of the moments.

1 Introduction

The problem of quantum scattering by chaotic or disordered systems is encountered in many fields ranging from atomic or molecular physics as well as in the scattering of electromagnetic microwaves. Some properties of the scattering process are well captured through the concept of time delay. This quantity, which goes back to Eisenbud and Wigner [1], is related to the time spent in the interaction region by a wave packet of energy peaked at E. It can be expressed in terms of the derivative of the S matrix with respect to the energy. In the context of chaotic scattering the approach based on random matrix theory (RMT) provides a statistical description of the time delays. This problem was first studied by a supersymmetric approach [2] and in [3] by using a statistical analysis. This latter work provides a derivation for the one channel case for the different universality classes. Recently it served as a starting point for [4] where the N channel distribution is shown to be given by the Laguerre ensemble of RMT. In spite of its success, such a description by RMT is not entirely satisfactory, in particular it does not apply to strictly one-dimensional systems [5] for which strong localisation effects occur. Furthermore it does not shed much light on the physical mechanisms which are responsible for the universal distribution. In this work we explore another approach by considering the scattering by a one-dimensional random potential. In this case the existence of universal distributions was first conjectured in [6] on the basis of a comparative study of two different models. This was further supported by [7] where the random potential is still of a different kind.

2 Universality of Wigner time delay distribution

The purpose of this section is to briefly recall a derivation [8] that accounts for the universality of the time delay distribution. We consider the Schrödinger equation on the half line $x \geqslant 0$:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_k(x) + V(x)\psi_k(x) = k^2\psi_k(x). \tag{1}$$

We assume that $V(x)$ has its support on the interval $[0, L]$ and impose the Dirichlet boundary condition $\psi_k(0) = 0$. Therefore, for $x \geqslant L$ stationary scattering states of the form

$$\psi_k(x) = \frac{1}{2}\left(\mathrm{e}^{-\mathrm{i}k(x-L)} + \mathrm{e}^{\mathrm{i}k(x-L)+\mathrm{i}\delta(k)}\right) \tag{2}$$

represent the superposition of an incoming and a reflected plane wave. Since there is only backward scattering, the reflection coefficient $\mathrm{e}^{\mathrm{i}\delta(k)}$ is of unit modulus and the Wigner time delay takes the form $\tau(k) \stackrel{\mathrm{def}}{=} \frac{1}{2k}\frac{\mathrm{d}\delta(k)}{\mathrm{d}k}$. Such a model with a random potential can be viewed as a model of a disordered sample connected to an infinite lead. Instead of using the invariant embedding method as in [9] or stochastic differential equations [10], our starting point is to relate the time delay to the wave function inside the sample. This may be achieved by using the so-called Smith formula[11] $\tau(k)=\frac{2}{k}\int_0^L \mathrm{d}x\,|\psi_k(x)|^2-\frac{1}{2k^2}\sin\delta(k)$. It expresses the time delay as the sum of a dwell time[12] and a term that can be neglected in the high energy limit. Inside the sample, the wave function may always be written as the product of an oscillatory term with some envelope. In the high energy (weak disorder) limit, it is possible to show that the oscillatory part presents rapid oscillations on a scale k^{-1}, whereas the envelope is a slow variable evolving on a scale λ (the localisation length) which allows to average over the rapid variable. Using the statistical properties of the envelope[13] it is possible to prove the following identity in law:

$$\tau(k) \overset{\text{(law)}}{=} \frac{\lambda}{k}\int_0^{L/\lambda} \mathrm{d}u\, \mathrm{e}^{-2u+2W(u)}\ , \tag{3}$$

where $W(x)$ is a normalised Wiener process ($\langle W(x)\rangle=0$, $\langle W(x)W(x')\rangle=\min(x,x')$). This representation of the time delay as an exponential functional of the Brownian motion[14,15,16] allows to derive a number of interesting results:

(i) the distribution[15]

$$P(\tau;L)=\frac{\lambda}{2k\tau^2}\mathrm{e}^{-\frac{\lambda}{2k\tau}}+\frac{2}{\pi\tau}\mathrm{e}^{-\frac{\lambda}{4k\tau}}\int_0^\infty ds\,\mathrm{e}^{-\frac{L}{2\lambda}(1+s^2)}\frac{s}{1+s^2}\,\mathrm{sh}\,\frac{\pi s}{2}\ \ W_{1,\frac{is}{2}}\left(\frac{\lambda}{2k\tau}\right) \tag{4}$$

($W_{\mu,\nu}(z)$ is a Whittaker's function) possesses a limit (τ fixed, $L\to\infty$) with an algebraic tail:

$$P(\tau)=\frac{\lambda}{2k\tau^2}\mathrm{e}^{-\frac{\lambda}{2k\tau}}\ . \tag{5}$$

(ii) Linear divergence of the first moment and exponential divergence of the higher moments[15]:

$$\langle\tau(k)\rangle=\frac{L}{k} \tag{6}$$

$$\langle\tau(k)^n\rangle=\left(\frac{-\lambda}{2k}\right)^n\left\{\sum_{m=2}^{n}(-1)^m C_n^m\frac{(m-2)!\,(2m-1)}{(n+m-1)!}\mathrm{e}^{2m(m-1)L/\lambda}-\frac{1}{n!}\left(2n\frac{L}{\lambda}+n-1\right)\right\} \tag{7}$$

In[6] we have shown that these results hold for two different models of random potential in the localised regime ($L\gg\lambda$).

The exact distribution $P(\tau;L)$ in the limit $L\to\infty$ (5) displays an algebraic tail, if however instead of considering as before the regime τ fixed $L\to\infty$ which leads to (5), one considers for fixed L the tail of the distribution in the limit $\tau\to\infty$, the distribution exhibits the following log-normal tail

$$P(\tau;L)\underset{\tau\to\infty}{\sim}\exp-\frac{\lambda}{8L}\ln^2\tau\ . \tag{8}$$

We were able to derive this behaviour when τ is in the range $\frac{\lambda}{2k}\mathrm{e}^{\sqrt{L/\lambda}}\ll\tau\ll\frac{\lambda}{2k}\mathrm{e}^{L/\lambda}$. However the fact that the most divergent part of the moments grows like $\mathrm{e}^{2n^2L/\lambda}$ suggests that the distribution is still log-normal for $\frac{\lambda}{2k}\mathrm{e}^{L/\lambda}\ll\tau$.

3 An heuristic argument to recover the moments

The derivation of the statistical properties of τ given above allows one to understand the universality of the result but on the other hand does not shed much light on the physical mechanisms which are responsible for the occurence of an algebraic tail. In the following we propose a physical picture based on the existence of resonances that explains the leading exponential behaviour of the moments. The starting point is to realize that the reflection of the incident wave on the random potential can in fact be viewed as a resonance tunneling process[17,18]. Indeed there exists

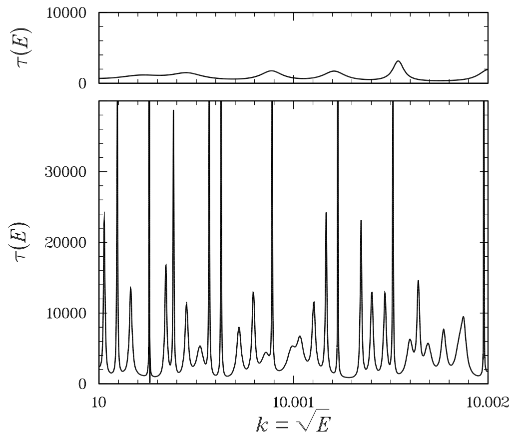

Figure 1: numerical computation (see section 4) of the time delay as a function of the energy for a given configuration of the disorder. Up: $L \simeq 0.12\lambda$. Down: $L \simeq 0.6\lambda$. When L is increased, some very narrow peaks appear. Those sharp resonances correspond to long lived states localised in the disordered region.

a representation of the time delay as a superposition of resonances of energy E_α and width Γ_α in the form [19]

$$\tau(E) = 2\sum_\alpha \frac{\Gamma_\alpha/2}{(E-E_\alpha)^2+\Gamma_\alpha^2/4}. \tag{9}$$

Obviously the dominant contribution $\tau \simeq \frac{4}{\Gamma_\alpha}$ is achieved when E is in a window of width Γ_α centered at E_α, and this will occur with probability $\frac{\Gamma}{\Delta}$ where Δ is the mean level spacing. In order to estimate the width we may assume that a discrete level E_α localised at x_0 will be broadened by its coupling to the continuum of states through the end point $x = L$. We may therefore set $\Gamma \sim \mathrm{e}^{-2\gamma_L(L-x_0)}$ where γ_L is the Lyapunov exponent in the finite system. Assuming that x_0 is uniformily distributed on $[0, L]$ and decorrelated from γ_L one obtains the estimate:

$$\langle \tau^n \rangle \sim \int_0^L \frac{\mathrm{d}x_0}{L} \int \mathrm{d}\gamma_L\, p(\gamma_L) \frac{\Gamma}{\Delta}\frac{1}{\Gamma^n}. \tag{10}$$

By using the distribution of the finite size Lyapunov exponent γ_L: $p(\gamma_L) = \sqrt{\frac{L}{2\pi\gamma}}\mathrm{e}^{-\frac{L}{2\gamma}(\gamma_L-\gamma)^2}$, one finally obtains

$$\langle \tau^n \rangle \sim \mathrm{e}^{2n(n-1)L/\lambda}. \tag{11}$$

This demonstrates that this particular behaviour of the moments has his origin both in the exponentially small widths of the resonances and in the fluctuations of the Lyapunov exponent for the finite size sample.

4 Numerical results

In order to test the analytical results in the above mentioned regime it is convenient to choose a model suitable for numerical simulations. For this purpose we have considered the case where the random potential is given by a sum of delta functions of the same weight v, randomly dropped on the half line with an average density ρ (the so-called Frish and Lloyd model)[a]. The equations that give the phase shift and the time delay then take a discrete form which can be conveniently implemented in a numerical simulation. The distribution of the time delay that we

[a] This model coincides with the Gaussian model (Halperin model) considered in [6] in the limit of a high density of impurities ($v \ll k \ll \rho$).

have obtained numerically in this way is in perfect agreement with equation (5) as soon as the high energy regime is reached. For example we compare on figure 2 the analytical expression (5) with the corresponding numerical result. The simulation was carried out at high energy ($k = 10$) for a system with 10^5 impurities of weight $v = 1$, distributed with an averaged density $\rho = 0.1$. The numerical calculation is based on a statistics of 50000 values. The ratio $L/\lambda = 125$ is sufficiently large for the limit distribution to be reached. It shows that the algebraic tail is well reproduced by (5) for 2000 times the typical value $\tau_{\mathrm{typ}} = 200$. Let us stress that there is no adjustable parameter to fit the numerics. The only parameter entering in the analytical expressions is the localisation length which is known for each kind of disorder.

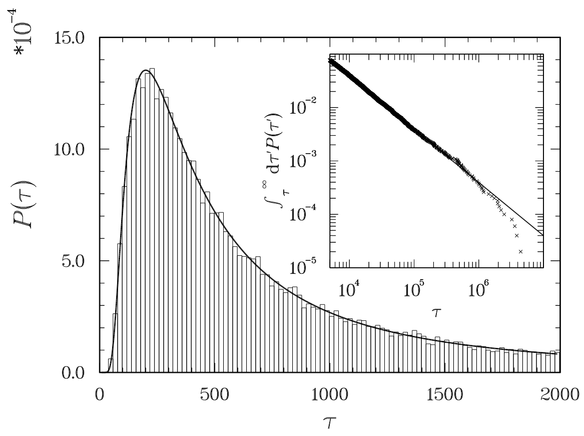

Figure 2: Time delay distribution in the localised regime $L \gg \lambda$. Comparison between the numerical calculation and expression (5). In the inset: tail of the integrated distribution, numerical and analytical.

Until now we have only considered a localised regime for which the size of the system is large compared to the localisation length $L \gg \lambda$. Another interesting case is the ballistic one characterised by $L \ll \lambda$. In this situation it was shown [8] that the time delay distribution is Gaussian with small fluctuations: $\langle\tau\rangle = \frac{L}{k}$ and $\langle\tau^2\rangle - \langle\tau\rangle^2 \simeq \frac{4}{3\lambda k^2} L^3$. In figure 3 we compare the numerical result and the Gaussian distribution. The calculation was performed in a regime $v \ll k \ll \rho$ and $(k^2 - \rho v) \gg (\rho v^2)^{2/3}$. In this case one has to take into account the mean value of the random potential. This implies that k has to be replaced by $\sqrt{k^2 - \rho v}$. The localisation length is thus $\lambda = \frac{8(k^2 - \rho v)}{\rho v^2}$. The calculation is performed for a ratio $L/\lambda \simeq 1.4\ 10^{-3}$. 10000 values of τ were calculated.

5 Final remarks

In this short note we have limited ourselves to the high energy or weak disorder regime for which the statistical properties of the time delay were shown to be universal. The statistical properties of the time delay for the low energy regime will be discussed elsewhere [8]. If one consider the model with δ scatterers randomly dropped in the low energy regime ($k \ll \rho, v$), it is possible to study numerically and in some regime ($k \ll \rho \ll v$) analytically the time delay distribution [8]. In this case the distribution exhibits an exponential tail. Therefore in the low energy regime the time delay distribution is no more given by the universal behaviour derivated above.

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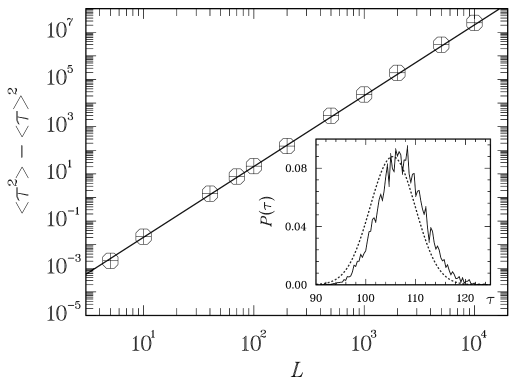

Figure 3: Second cumulant of the time delay in the ballistic regime $L \ll \lambda$. Comparison between numerical results and the analytical result given in the text. The parameters are $v = 0.001$, $\rho = 100$ and $k = 1$. In the inset: time delay distribution for $L = 100$ (10^4 impurities).

of figures.

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