Wigner Time Delay Distribution for One-dimensional Random Potentials

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Ref. : J. Phys. A : Math. Gen. **30**, 8017, (1997) $\longrightarrow$
Phys. Rev. Lett. **82**, 4220, (1999)

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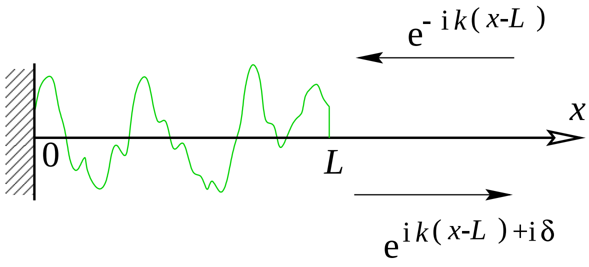
1. Introduction



Phase shift : $\delta(E)$

Wigner time delay : $\boxed{\tau(E) = \frac{\mathrm{d}}{\mathrm{d}E}\delta(E)}$

Eisenbud, '48

Wigner, '55

for many scattering channels : Smith, '60

Random Matrix Theory Approach :

- diffusive regime ($\neq$ 1D systems)
- universality of $P(\tau)$

$$P(\tau) \propto \frac{\mathrm{e}^{-t_0/\tau}}{\tau^{2+\eta}}$$

where t_0 and η are functions of the number of channels N and the universality class β (=1,2 ou 4)

Fyodorov & Sommers, '96

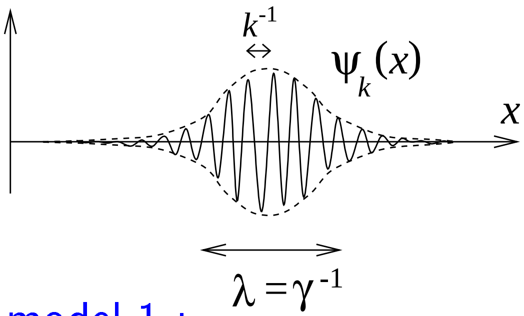
Gopar, Mello & Büttiker, '96

Brouwer, Frahm & Beenakker, '97

$\rightarrow$ Universality of $P(\tau)$ for 1D disordered systems ?

2. First approach

$\longrightarrow$ for one-dimensional systems : all states are localised



λ : localisation length
γ : Lyapunov exponent

$k^2 = E \gg \text{disorder} \Rightarrow k^{-1} \ll \lambda$

model 1 :

$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$

$V(x)$ is gaussian $\Rightarrow \left\{ \begin{array}{l} \langle V(x) \rangle = 0 \\ \big\langle V(x)V(x') \big\rangle = \sigma\delta(x-x') \end{array} \right.$

$$\gamma(E) \underset{E \gg \sigma^{2/3}}{\simeq} \frac{\sigma}{8E}$$

model 2 : Supersymmetric disordered quantum mechanics.

$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 + \phi'(x) = Q^\dagger Q$ where $Q = -\frac{\mathrm{d}}{\mathrm{d}x} + \phi(x)$

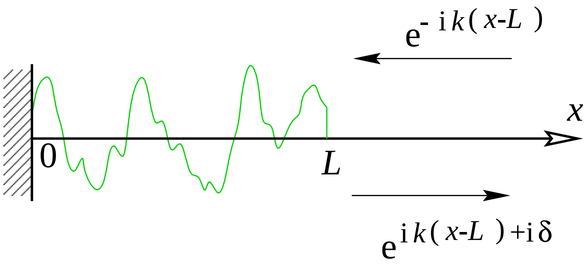
$\phi(x)$ is gaussian $\Rightarrow \left\{ \begin{array}{l} \langle \phi(x) \rangle = 0 \\ \big\langle \phi(x)\phi(x') \big\rangle = g\delta(x-x') \end{array} \right.$

$$\gamma(E) \underset{E \gg g^2}{\simeq} \frac{g}{2}$$

$\longrightarrow$ these two models exhibit different spectral and localisation properties.

Scattering states :

$$\psi_k(x) = \mathrm{e}^{-\mathrm{i}k(x-L)} + \mathrm{e}^{\mathrm{i}k(x-L)+\mathrm{i}\delta(k)} \text{ for } x \geqslant L$$



$$E = k^2$$

$H\psi_k = k^2\psi_k \Rightarrow$ coupled stochastic differential equations (SDE) for δ and τ

$\rightarrow$ those SDE may be related to a Fokker-Planck equation (FPE) for $P(\delta, \tau; L)$

At high energy :

$\rightarrow$ δ is uniformly distributed

$\rightarrow$ one may assume decorrelation of δ and τ.

$\Rightarrow$ FPE equations for $P(\delta, \tau; L)$ are different for models 1 and 2 but average over δ leads for both models to

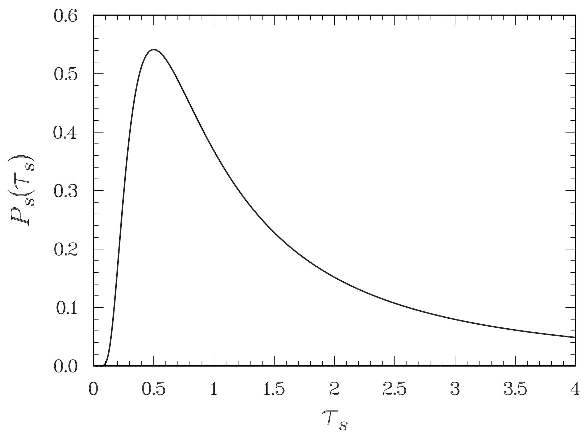
$$\frac{\partial}{\partial L} P(\tau; L) = 2\gamma \left(\frac{\partial}{\partial \tau} \tau \frac{\partial}{\partial \tau} \tau + \frac{\partial}{\partial \tau} \left(\tau - (2k\gamma)^{-1} \right) \right) P(\tau; L)$$

$\Rightarrow$ Same distribution of τ for both models

a) Localised regime ($L \gg \lambda$)

$\rightarrow$ Existence of a limit distribution ($L \rightarrow \infty$)

$$P_s(\tau_s) = \frac{1}{\tau_s^2} \mathrm{e}^{-1/\tau_s}$$



$\tau_s = \frac{\tau}{\lambda/2k}$

$\rightarrow$ moments for finite L :

$$\begin{aligned} \langle \tau \rangle &= \frac{L}{k} \\ \langle \tau^n \rangle &\simeq \frac{(n-2)!}{(2n-2)!} \left(\frac{\lambda}{2k} \right)^n \mathrm{e}^{2n(n-1)\frac{L}{\lambda}} \end{aligned}$$

Monthus & Comtet, '94

b) Ballistic regime ($L \ll \lambda$)

$\rightarrow$ $P(\tau)$ is gaussian with small fluctuations

$\rightarrow$ moments :

$$\begin{aligned} \langle \tau \rangle &\simeq \frac{L}{k} \\ \langle \tau^2 \rangle - \langle \tau \rangle^2 &\simeq \frac{4}{3\lambda k^2} L^3 = \frac{4}{3} \left(\frac{L}{k} \right)^2 \frac{L}{\lambda} \ll \langle \tau \rangle^2 \end{aligned}$$

3. Universality of $P(\tau)$

$\psi_k(x) = \frac{1}{2}(\mathrm{e}^{-\mathrm{i}k(x-L)} + \mathrm{e}^{\mathrm{i}k(x-L)+\mathrm{i}\delta(k)})$ for $x \geqslant L$

$\psi_k(x) = \mathcal{N}\mathrm{e}^{\xi(x)} \sin\theta(x)$ for $x \leqslant L$

$H\psi_k = k^2\psi_k \Rightarrow$

$$\tau(k) = \frac{1}{k}\int_0^L \mathrm{d}x\, 2|\psi_k(x)|^2 - \frac{1}{2k^2}\sin\delta(k)$$

Smith, '60

$\rightarrow$ At high energy : $\sin\theta(x)$ oscillates on a length scale k^{-1}
$\mathrm{e}^{\xi(x)}$ varies on a length scale $\lambda \gg k^{-1}$

$\Rightarrow \tau(k) = \frac{1}{k}\int_0^L \mathrm{d}x\, \mathrm{e}^{2(\xi(x)-\xi(L))}$

$$\xi(x) \overset{\text{(loi)}}{=} \gamma x + \sqrt{\gamma}W(x)$$

$W(x)$ is a normalised Wiener process : $\langle W(x)\rangle = 0$ and $\langle W(x)W(x')\rangle = \min(x, x')$ (Antsygina, Pastur & Slyusarev, '81)

$$\boxed{\tau(k) \overset{\text{(law)}}{=} \frac{\lambda}{k}\int_0^{L/\lambda} \mathrm{d}u\, \mathrm{e}^{-2u+2W(u)}}$$

$\rightarrow$ exponential functional of the Brownian motion (studied by mathematicians : Yor, '92). Those objects also arises in the context of 1D classical diffusion in random media (Bouchaud, Comtet, Georges & le Doussal '90).

$\rightarrow$ contains all previous statistical properties

4. Numerical results

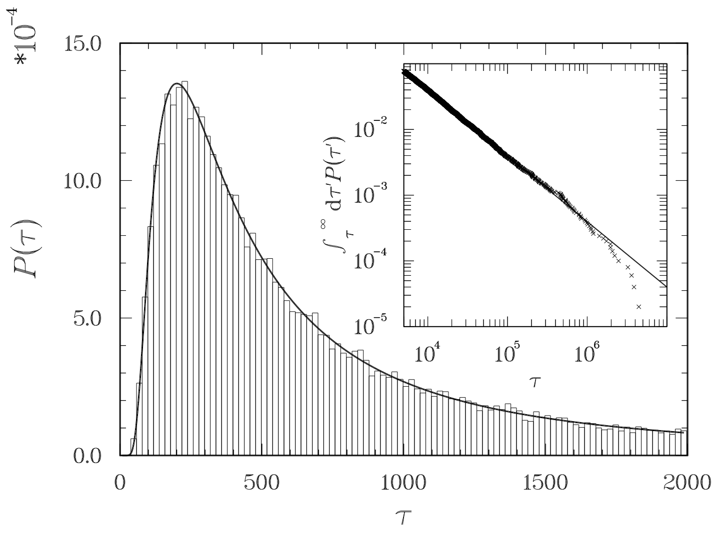
modèle 3 : $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$

where $V(x) = v \sum_i \delta(x - x_i)$

$\{x_i\}$: Poisson distribution with average density ρ

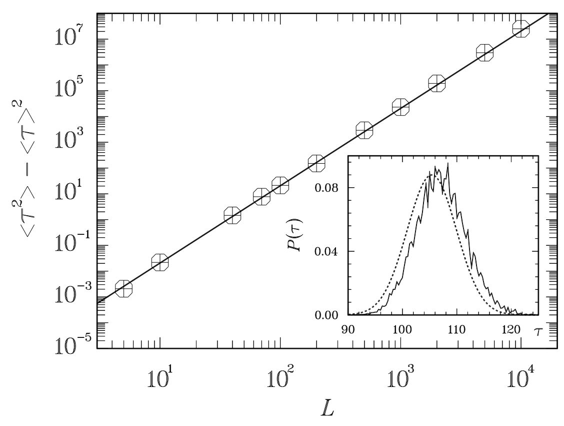
$\rightarrow$ equations for δ and τ are discrete

a) Localised regime ($L \gg \lambda$)



$k = 10$
$v = 1$
$\rho = 0.1$
$N = 100000$ impurities
$n_s = 50000$ (statistics)

b) Ballistic regime ($L \ll \lambda$)



$$\langle \tau^2 \rangle - \langle \tau \rangle^2 = \frac{\rho v^2}{6(k^2 - \rho v)^2} L^3$$

$k = 1$
$v = 0.001$
$\rho = 100$

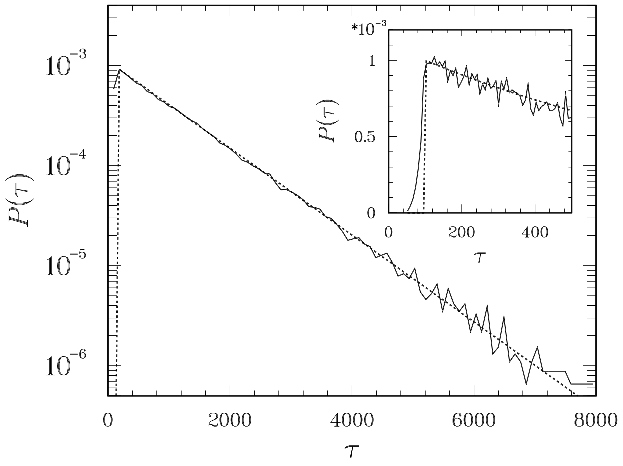
inset :
$L/\lambda \simeq 1.4\ 10^{-3}$
$n_s = 10000$

<u>Low energy :</u>

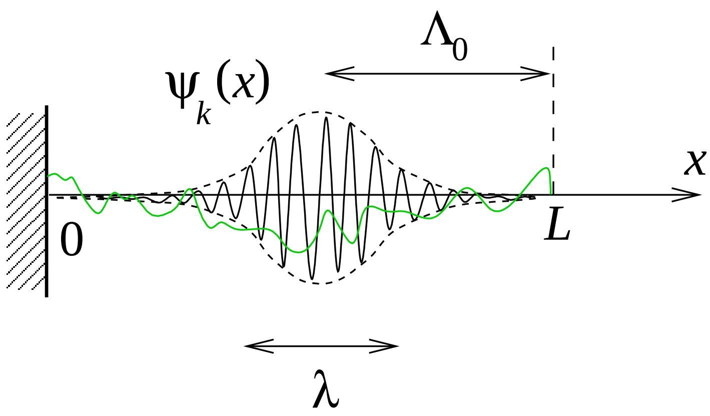
if $k \ll \rho \ll v$

$\rightarrow P(\delta)$ is non uniform

$\rightarrow P(\tau) \simeq k\rho\, \Upsilon(\tau - \frac{1}{kv})\, \mathrm{e}^{-k\rho(\tau - \frac{1}{kv})}$



5. A physical picture for the localised regime

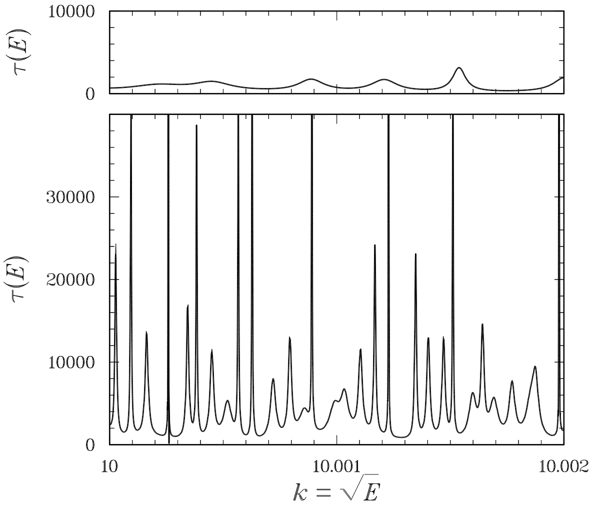


States localised in the disorder region are coupled to the continuum

• Time delay for two configurations of the disorder :

a) strong coupling to the continuum : $L \simeq 0.12\lambda$

b) weak coupling to the continuum : $L \simeq 0.6\lambda$



$$\tau(E) = \sum_\alpha \frac{\Gamma_\alpha}{(E - E_\alpha)^2 + \Gamma_\alpha^2/4}$$

Fyodorov & Sommers, '96

$\rightarrow$ moments :

dominant contributions to $\langle \tau^n \rangle$ come from $E \simeq E_\alpha \Rightarrow \tau \sim \frac{1}{\Gamma_\alpha}$

resonance width : $\Gamma \sim \mathrm{e}^{-2\gamma_L \Lambda_0}$

mean spacing between resonances peaks : $\Delta \simeq \frac{2\pi\sqrt{E}}{L}$

$$\langle \tau^n \rangle \sim \int_0^L \frac{\mathrm{d}\Lambda_0}{L} \int \mathrm{d}\gamma_L \, p(\gamma_L) \frac{\Gamma}{\Delta} \frac{1}{\Gamma^n}$$
$$\sim \mathrm{e}^{2n(n-1)L/\lambda}$$

$\rightarrow$ tail of the distribution :

$$P(\tau, L) \underset{\tau\to\infty}{\sim} \mathrm{e}^{-\frac{1}{8\gamma L}\ln^2 \tau}$$