# Out of equilibrium statistical physics – Exam

Wednesday 7 january 2015

Write your answers to this part of the exam on a **SEPARATE** sheet.
Vous pouvez rédiger en français si vous le souhaitez.

Subject : **Weiss oscillations of the magnetoresistivity in a 2DEG**

**Introduction :** Gerhardts, Weiss and von Klitzing analysed the conductivity of a **two dimensional** electron gas (electrons confined at the interface of two semiconductors) submitted to a strong magnetic field. They were able to introduce some additional external (scalar) potential modulated in space in one direction only, $V(x) = V_0 \cos(2\pi x/a)$. The aim of the problem is to study the effect of $V(x)$ on the longitudinal conductivity.

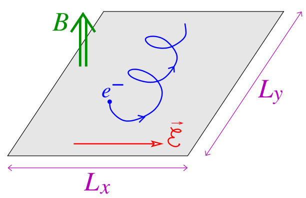
The Hamiltonian describing the dynamics of **one** electron moving in the plane $xOy$ is

$$H_0 = \frac{1}{2} m_* \vec{v}^{\,2} + V(x) = \frac{\left(\vec{p} - e\vec{A}(\vec{r})\right)^2}{2m_*} + V(x) \qquad (1)$$

where $\vec{v} = (v_x, v_y)$ is the velocity operator and $\vec{A} = (0, B\,x, 0)$ describes a magnetic field perpendicular to the plane. $m_* = 0.067\, m_e$ is the effective mass in AsGa ($m_e \simeq 10^{-30}$ kg).

We denote by $\{\varepsilon_\alpha, \,|\varphi_\alpha\rangle\}$ the spectrum of $H_0$.

**A. Conductivity.–** The conductivity tensor characterizes the response of the spatially averaged current density $\vec{j}$ to an external uniform time dependent electric field :




$$\langle j_i(t) \rangle_{\vec{\mathcal{E}}} = \sum_j \int \mathrm{d}t'\, \sigma_{ij}(t-t')\, \mathcal{E}_j(t') + \mathcal{O}(\mathcal{E}^2)\,. \qquad (2)$$

The current density operator for one electron is $\vec{j} = \frac{e}{\mathrm{Surf}} \vec{v}$, where $\mathrm{Surf} = L_x L_y$ is the surface of the plane. The perturbation is chosen under the form $H_{\mathrm{pert}}(t) = -e\,\vec{\mathcal{E}}(t)\cdot\vec{r}$.

1. Express the conductivity tensor as an equilibrium correlation function (for one electron).
2. Show that the frequency dependent conductivity for the electron **gas** is given by

$$\widetilde{\sigma}_{ij}(\omega) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\, \sigma_{ij}(t) = -\frac{\mathrm{i}\hbar e^2}{\mathrm{Surf}} \sum_{\alpha,\beta} \frac{f_\alpha - f_\beta}{\varepsilon_\alpha - \varepsilon_\beta} \frac{(v_i)_{\alpha\beta}(v_j)_{\beta\alpha}}{\hbar\omega + \varepsilon_\alpha - \varepsilon_\beta + \mathrm{i}\,0^+}\,, \qquad (3)$$

   where $f_\alpha \equiv f(\varepsilon_\alpha)$ is the Fermi function.

   Hint : use the formula recalled in the appendix.

3. Discuss the role and the origin of the « $\mathrm{i}\,0^+$ » in the denominator.
4. **Zero frequency.–** We will consider the zero frequency conductivity $\sigma_{ij} \equiv \widetilde{\sigma}_{ij}(\omega = 0)$ :

$$\sigma_{ij} = -\frac{\mathrm{i}\hbar e^2}{\mathrm{Surf}} \sum_{\alpha,\beta} \frac{f_\alpha - f_\beta}{\varepsilon_\alpha - \varepsilon_\beta} \frac{(v_i)_{\alpha\beta}(v_j)_{\beta\alpha}}{\varepsilon_\alpha - \varepsilon_\beta + \mathrm{i}\hbar/\tau}\,, \qquad (4)$$

   where we performed the substitution $0^+ \longrightarrow \hbar/\tau$. What is the physical meaning of $\tau$ ?

   We admit that the formula is meaningfull for a finite $1/\tau$.

### B. Conductivity of the Landau problem (case $V(x) = 0$).

**The Landau problem.**– The one body Hamiltonian has the explicit form

$$H_0 = \frac{1}{2} m_* \vec{v}^{\,2} = \frac{1}{2m_*} p_x^2 + \frac{1}{2} m_* \omega_c^2 \left( x - \frac{p_y}{eB} \right)^2 \tag{5}$$

where $\omega_c = \frac{eB}{m_*} > 0$ is the cyclotron pulsation. Translation invariance in the $y$ direction allows to write the eigenstates as $\varphi(x, y) = f(x)\, \mathrm{e}^{\mathrm{i}ky}$. We obtain that $f(x)$ is the eigenstate for a 1D harmonic oscillator centered around $x_c = \hbar k/eB$. Denoting by $\phi_n(x)$ the well-known eigenfunctions of the harmonic oscillator (Gaussian×Hermite polynomial), we find that the eigenstates of $H_0$ are

$$\varphi_{n,x_c}(x, y) = \phi_n(x - x_c) \frac{\mathrm{e}^{\mathrm{i}x_c y/\ell_B^2}}{\sqrt{L_y}} \qquad \text{for an energy } \varepsilon_n = \hbar\omega_c \left( n + \frac{1}{2} \right) , \qquad n \in \mathbb{N} . \tag{6}$$

$\ell_B = \sqrt{\hbar/(eB)}$ is the magnetic length setting the typical width of the narrow function $\phi_n(x)$. The quantum number $x_c$ replaces the wavevector $k$. Imposing periodic boundary condition in the $y$ direction, we obtain that $x_c$ is quantised as $x_c = 2m\pi\ell_B^2/L_y$, with $m \in \mathbb{N}$. The spectrum of the one particle Hamiltonian is similar to that of a harmonic oscillator (reflecting the existence of the cyclotron orbits) with **macroscopically degenerate** Landau levels (because $\varepsilon_n$ is independent on quantum number $x_c$) : degeneracy of levels is $d_{\mathrm{LL}} = L_x L_y/(2\pi\ell_B^2)$, which follows from $x_c \in [0, L_x]$.

Matrix elements of $\vec{v}$ are deduced from standard properties of the 1D harmonic oscillator

$$\langle\, \varphi_{n,x_c} | v_x | \varphi_{m,x_c'} \rangle = -\mathrm{i}\, \delta_{x_c,x_c'} \sqrt{\frac{\hbar\omega_c}{2m_*}} \left( \sqrt{n+1}\, \delta_{m,n+1} - \sqrt{n}\, \delta_{m,n-1} \right) \tag{7}$$

$$\langle\, \varphi_{n,x_c} | v_y | \varphi_{m,x_c'} \rangle = -\delta_{x_c,x_c'} \sqrt{\frac{\hbar\omega_c}{2m_*}} \left( \sqrt{n+1}\, \delta_{m,n+1} + \sqrt{n}\, \delta_{m,n-1} \right) \tag{8}$$

1. Compute explicitly the longitudinal (zero frequency) conductivity $\sigma_{xx} = \sigma_{yy}$ from Eq. (4). Express the result in terms of the Drude conductivity $\sigma_0 = \frac{n_e e^2 \tau}{m_*}$, where $n_e = N/\mathrm{Surf}$ is the electronic density, and a dimensionless function of $\omega_c\tau$. Plot neatly $\sigma_{yy}$ as a function of the magnetic field and explain physically the behaviour.

   Hint : the filling factor (number of filled Landau levels) may be written as $N/d_{\mathrm{LL}} = n_e h/(eB) = \sum_{n=1}^{\infty} n \left[ f(\varepsilon_{n-1}) - f(\varepsilon_n) \right]$

2. One could compute the Hall conductivity along the same lines : one obtains $\sigma_{xy} = -\sigma_{yx} = \sigma_0\, \omega_c\tau / \left[ 1 + (\omega_c\tau)^2 \right]$. Deduce the resistivity tensor $\rho = \sigma^{-1}$.

**C. Effect of the oscillating potential $V(x)$.**– We now consider the experimental situation described by the Hamiltonian (1) where $V(x) = V_0 \cos(2\pi x/a)$.

1. If the modulation occurs on large scale, $a \gg \ell_B$, justify that (6) are still eigenstates of the Hamiltonian, with eigenvalues now depending on the quantum number $x_c$ as

$$\varepsilon_{n,x_c} \simeq \hbar\omega_c \left( n + \frac{1}{2} \right) + V(x_c) \, . \tag{9}$$

2. Show that the speed operator in the $y$ direction now aquires some non zero diagonal matrix element, given by the Feynman-Hellmann theorem :

$$\langle\, \varphi_{n,x_c} | v_y | \varphi_{n,x_c'} \rangle = \delta_{x_c,x_c'} \frac{1}{m_*\omega_c} \frac{\partial \varepsilon_{n,x_c}}{\partial x_c} \, . \tag{10}$$

   Hint : write $v_y = \omega_c(x_c - x)$.

In the expression of the conductivity (4), considering separatly the diagonal terms $\sum_{\alpha=\beta}$ and non diagonal term $\sum_{\alpha\neq\beta}$ [the indices designate each a pair of quantum numbers $\alpha \equiv (n, x_c)$], leads to the conclusion that the longitudinal conductivity receives an additional contribution $\sigma_{yy} \longrightarrow \sigma_{yy} + \Delta\sigma_{yy}$ due to the introduction of the oscillating potential :

$$\Delta\sigma_{yy} = \frac{e^2\tau}{\text{Surf}} \sum_{n,\, x_c} -f'(\varepsilon_{n,x_c})\, |\langle\, \varphi_{n,x_c} | v_y | \varphi_{n,x_c} \,\rangle|^2 \qquad (11)$$

3. Is the conductivity $\sigma_{xx}$ affected by the presence of $V(x)$ ? Justify your answer.
4. Bonus : Assuming that $\hbar\omega_c > V_0$, show that the zero temperature result reads

$$\Delta\sigma_{yy} = \frac{e^2}{h} \frac{2\tau\ell_B^2}{a\,\hbar} |V'(X_n)| \qquad \text{where } V(X_n) = \varepsilon_F - \hbar\omega_c(n + 1/2)\ . \qquad (12)$$

   Hint : $\sum_{x_c} \longrightarrow \frac{L_y}{2\pi\ell_B^2} \int_0^{L_x} \mathrm{d}x_c$

5. The expression (12) does not allow for a simple analysis of the experiment for two reasons : $(i)$ one should add several such contributions when $\hbar\omega_c < V_0$ ($0 < \hbar\omega_c < 1.2$ meV and $V_0 = 0.3$ meV) and $(ii)$ thermal effect is not negligible ($k_BT = 0.19$ meV) ; many Landau levels are filled ($\varepsilon_F = 11$ meV). Recalling that the longitudinal transport involves cyclotron orbits of energy $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m_*}$, i.e. of radius $R_c = k_F\ell_B^2$, argue that one expects that $\Delta\sigma_{yy}$ is (pseudo) periodic as a function of $1/B$. Give the period of these oscillations.

   Explain qualitatively the data of Fig. 1 in the low magnetic field domain ($B \lesssim 0.5$ T).

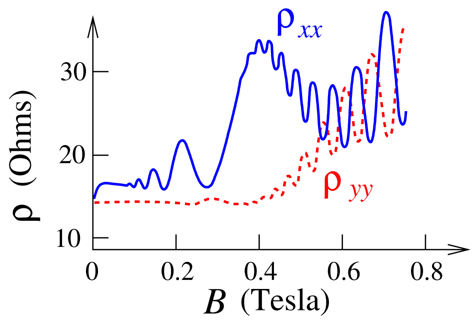



Figure 1 – *Magneto-resistivities $\rho_{xx}$ (continuous) and $\rho_{yy}$ (dashed) at $T = 2.2$ K (i.e. $k_BT = 0.19$ meV). Data from : Gerhardts, et al, Phys. Rev. Lett.* ***62****, 1173 (1989).*

6. Bonus : What is the origin of the rapid oscillations appearing at larger field ($B \gtrsim 0.4$T) ?

☞ Appendix :

We recall that the grand canonical average of a commutator of many body operators, sums of one particle operators, of the form $\hat{A} = \sum_{i=1}^N \hat{a}^{(i)}$ and $\hat{B} = \sum_{i=1}^N \hat{b}^{(i)}$, is

$$\langle[\hat{A}\,, \hat{B}]\rangle = \sum_\alpha f_\alpha \,\langle\, \varphi_\alpha \,|\, [\hat{a}, \hat{b}] \,|\, \varphi_\alpha \,\rangle = \sum_{\alpha,\,\beta} (f_\alpha - f_\beta)\, a_{\alpha\beta}\, b_{\beta\alpha}\ , \qquad (13)$$

where the sum runs over one particle stationary states. $a_{\alpha\beta} = \langle\, \varphi_\alpha \,|\hat{a}|\, \varphi_\beta \,\rangle$ is a matrix element of the one particle operator and $f_\alpha = f(\varepsilon_\alpha)$ denotes the occupancy of the individual eigenstate.

Solutions à l'adresse `http://www.lptms.u-psud.fr/christophe_texier/`

**Weiss oscillations of the magnetoresistivity in a 2DEG**
SOLUTIONS

## A. Conductivity.

1. With the perturbation of the form $H_{\rm pert}(t) = -e\,\vec{\mathcal{E}}(t)\cdot\vec{r}$, we get the conductivity for one electron as $\sigma_{ij}(t) = \frac{\mathrm{i}}{\hbar}\,\theta_{\rm H}(t)\left\langle\left[\frac{e}{\rm Surf}v_i(t)\,,\,e\,r_j\right]\right\rangle$.

2. The conductivity of the gas of $N$ electrons is obtained by doing $v_i \to \sum_{a=1}^N v_i^{(a)}$ and $r_j \to \sum_{a=1}^N r_j^{(a)}$ in the correlator. Using the formula of the appendix for the grand canonical average of the commutator, we get

$$\sigma_{ij}(t) = \frac{\mathrm{i}e^2}{\hbar\,\rm Surf}\,\theta_{\rm H}(t)\,\mathrm{tr}\left\{f(H_0)\;[v_i(t)\,,\,r_j]\right\} \tag{14}$$

$$= \frac{\mathrm{i}e^2}{\hbar\,\rm Surf}\,\theta_{\rm H}(t)\,\sum_{\alpha,\,\beta}(f_\alpha - f_\beta)\,(v_i)_{\alpha\beta}\,\mathrm{e}^{\mathrm{i}(\varepsilon_\alpha-\varepsilon_\beta)t/\hbar}\,(r_j)_{\beta\alpha} \tag{15}$$

   where the trace, runing in the one electron Hilbert space, has been expanded over the one electron eigenstates ; $f_\alpha \equiv f(\varepsilon_\alpha)$ is the Fermi function.

   The last steps are :
   ($i$) Fourier transform $\int \mathrm{d}t\,\mathrm{e}^{+\mathrm{i}\omega t}$ and use $\int_0^\infty \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\Omega t} = \frac{1}{0^+ - \mathrm{i}\,\Omega}$, where $0^+$ is a regulator.
   ($ii$) Make use of $\vec{v} = \frac{\mathrm{i}}{\hbar}[H_0, \vec{r}]$ to relate the matrix elements $(r_j)_{\beta\alpha} = \mathrm{i}\hbar\,(v_j)_{\beta\alpha}/(\varepsilon_\alpha - \varepsilon_\beta)$. We get (3).

3. The $\mathrm{i}\,0^+$ is a *regulator* which shifts the pole away from the real axis of the frequency. It originates from the heaviside function, i.e. from the *causality* of the response.

4. The subtitution $0^+ \longrightarrow \hbar/\tau$ recalls the physical interpretation of the regulator as an infinitly small *damping rate* (here this should correspond to the relaxation of the momentum orientation due to collisions).

   The formula (4) will be the starting point of the analysis in the following, assuming that it is correct to consider a constant damping rate (this is the « constant relaxation rate approximation »).

## B. Conductivity of the Landau problem (case $V(x) = 0$).

1. The index $\alpha$ labelling the eigenstates is replaced by the quantum numbers $(n, x_c)$. Injecting the matrix element in (4) we get

$$\sigma_{yy} = -\frac{\mathrm{i}e^2\hbar}{\rm Surf}\sum_{n,\,x_c,\,m,\,x_c'}\frac{f_n - f_m}{\varepsilon_n - \varepsilon_m}\,\delta_{x_c,x_c'}\,\frac{\hbar\omega_c}{2m_*}\,\frac{\left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right)^2}{\varepsilon_n - \varepsilon_m + \mathrm{i}\hbar/\tau} \tag{16}$$

$$= -\frac{\mathrm{i}e^2\hbar}{\rm Surf}\,\frac{\hbar\omega_c}{2m_*}d_{\rm LL}\left(\sum_{n=0}^\infty\frac{f_n - f_{n+1}}{\varepsilon_n - \varepsilon_{n+1}}\,\frac{n+1}{\varepsilon_n - \varepsilon_{n+1} + \mathrm{i}\hbar/\tau} + \sum_{n=1}^\infty\frac{f_n - f_{n-1}}{\varepsilon_n - \varepsilon_{n-1}}\,\frac{n}{\varepsilon_n - \varepsilon_{n-1} + \mathrm{i}\hbar/\tau}\right)$$

$$= \frac{\mathrm{i}e^2\hbar}{\rm Surf}\,\frac{\hbar\omega_c}{2m_*}d_{\rm LL}\left(\frac{1}{-\hbar\omega_c + \mathrm{i}\hbar/\tau} + \frac{1}{\hbar\omega_c + \mathrm{i}\hbar/\tau}\right)\frac{1}{\hbar\omega_c}\,\underbrace{\sum_{n=1}^\infty n\left[f_{n-1} - f_n\right]}_{=N/d_{\rm LL}=n_e h/(eB)\ \text{(filling factor)}} \tag{17}$$

   Finally we obtain

$$\sigma_{yy} = \sigma_{xx} = \sigma_0\,\frac{1}{1 + (\omega_c\tau)^2} \tag{18}$$

where $\sigma_0 = \frac{n_e e^2 \tau}{m_*}$ is the Drude conductivity.

The magnetoconductivity has the form of a Lorentzian as a function of $B \propto \omega_c$. The decay of the longitudinal conductivity comes from the effect of the Lorentz force which bends the electronic trajectories.

2. The calculation of the Hall conductivity (not asked) follows the same lines. The product of matrix element is

$$\sigma_{yy} = \cdots \left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right)^2 \cdots$$
$$\stackrel{\text{replaced by}}{\longrightarrow} \quad \sigma_{xy} = \cdots \left(\sqrt{n+1}\,\delta_{m,n+1} - \sqrt{n}\,\delta_{m,n-1}\right)\left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right) \cdots$$

hence there is now a minus sign :

$$\sigma_{xy} = \cdots \left( -\frac{1}{-\hbar\omega_c + \mathrm{i}\hbar/\tau} + \frac{1}{\hbar\omega_c + \mathrm{i}\hbar/\tau} \right) \cdots \quad \Rightarrow \quad \sigma_{xy} = (\omega_c\tau)\,\sigma_{yy} = \sigma_0\,\frac{\omega_c\tau}{1+(\omega_c\tau)^2} \tag{19}$$

The inversion of the conductivity tensor reads :

$$\sigma = \sigma_0\,\frac{1}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \quad \Rightarrow \quad \rho = \frac{1}{\sigma_0}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix} \tag{20}$$

i.e. explicitly

$$\rho_{xx} = \rho_{yy} = \frac{1}{\sigma_0} \quad \text{and} \quad \rho_{yx} = -\rho_{xy} = \frac{B}{n_e e}\,. \tag{21}$$

**C. Effect of the oscillating potential $V(x)$.**– The complete Hamiltonian is now

$$H_0 = \frac{1}{2m_*}p_x^2 + \frac{1}{2}m_*\omega_c^2\left(x - \frac{p_y}{eB}\right)^2 + V(x)\,. \tag{22}$$

1. We can still look for eigenstates under the form $\varphi(x,y) = f(x)\,\mathrm{e}^{\mathrm{i}x_c y/\ell_B^2}$.

The low energy states of the quadratic potential ($V(x) = 0$) are spread over a distance $\sim \ell_B = \sqrt{\hbar/(eB)}$. If the potential $V(x)$ is smooth at this scale (for $a \gg \ell_B$), it may be considered as constant, replaced by its value at the minimum of the quadartic potential $\to V(x_c)$, hence the spectrum of energy

$$\varepsilon_{n,x_c} \simeq \hbar\omega_c\left(n + \frac{1}{2}\right) + V(x_c)\,. \tag{23}$$

The denegeracy of Landau levels is now lifted.

2. We write

$$\langle\,\varphi_{n,x_c}\,|\hat{v}_y|\,\varphi_{n,x_c}\,\rangle = \langle\,\varphi_{n,x_c}\,|\omega_c(x_c - \hat{x})|\,\varphi_{n,x_c}\,\rangle = \frac{1}{m_*\omega_c}\langle\,\varphi_{n,x_c}\,|\frac{\partial}{\partial x_c}\frac{1}{2}m_*\omega_c^2(x_c-\hat{x})^2|\,\varphi_{n,x_c}\,\rangle$$

We may now introduce the other terms of the Hamiltonian as they do not depend on $x_c$ :

$$\frac{1}{m_*\omega_c}\langle\,\varphi_{n,x_c}\,|\frac{\partial}{\partial x_c}\left[\frac{1}{2}m_*\hat{v}_x^2 + \frac{1}{2}m_*\omega_c^2(x_c-\hat{x})^2 + V(\hat{x})\right]|\,\varphi_{n,x_c}\,\rangle\,.$$

The partial derivative may be extracted from the quantum averaging thanks to normalisation condition $\frac{\partial}{\partial x_c}\langle\,\varphi_{n,x_c}\,|\,\varphi_{n,x_c}\,\rangle = 0$. Finally we obtain the desired relation (application of the Feynman-Hellmann theorem)

$$\langle\,\varphi_{n,x_c}\,|\hat{v}_y|\,\varphi_{n,x_c}\,\rangle = \frac{\partial\varepsilon_{n,x_c}}{\partial x_c} = \frac{V'(x_c)}{m_*\omega_c} \tag{24}$$

The fact that the Landau band is not flat anymore (degeneracy lifted) is related to the presence of a current in the $y$ direction.

Note that the current in the other direction is not affected : $\langle \varphi_{n,x_c} | \hat{v}_x | \varphi_{n,x_c} \rangle = 0$ (it follows from the effective confinment by the quadratic potential in the $x$ direction).

3. Off-diagonal matrix elements of the velocity operator are not changed by the introduction of the smooth $V(x)$.

   Hence the non vanishing diagonal matrix elements $\langle \varphi_{n,x_c} | \hat{v}_y | \varphi_{n,x_c} \rangle$ brings an additional contribution, $\sigma_{yy} \longrightarrow \sigma_{yy} + \Delta\sigma_{yy}$, to the longitundinal conductivity (4) :

$$\Delta\sigma_{yy} = \frac{e^2\tau}{\text{Surf}} \sum_\alpha -f'(\varepsilon_\alpha) \left| (v_y)_{\alpha\alpha} \right|^2 \tag{25}$$

   $\sigma_{xx}$ is not changed by the introduction of $V(x)$ since $(v_x)_{\alpha\alpha} = 0$.

4. We analyse the conductivity at $T = 0$. We assume $\hbar\omega_c > V_0$ for simplicity, which means that at most one oscillating Landau band may cross the Fermi level $\varepsilon_F$ (Fig. 2).

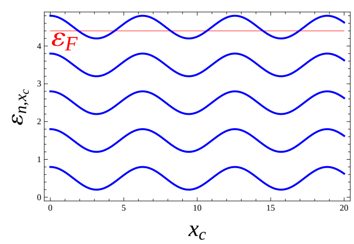



FIGURE 2 – *Landau bands.*

   We can write

$$\Delta\sigma_{yy} = \frac{e^2\tau}{L_x L_y} \frac{L_y}{2\pi\ell_B^2} \int_0^{L_x} \mathrm{d}x_c \sum_n \delta(\varepsilon_F - \varepsilon_{n,x_c}) \left| \frac{V'(x_c)}{m_*\omega_c} \right|^2 \tag{26}$$

   Using the periodicity of the potential : $\int_0^{L_x} \mathrm{d}x_c \longrightarrow \frac{L_x}{a/2} \int_0^{a/2}$ and that $\delta(\varepsilon_F - \varepsilon_{n,x_c}) = \delta(x_c - X_n)/|V'(X_n)|$, we obtain (12).

   We may be more explicit by using the expression of the cosine potential. We obtain finally

$$\Delta\sigma_{yy} = \frac{2e^2}{\hbar} \frac{\ell_B^2}{a^2} \sum_{n=0}^{\infty} \frac{\tau\sqrt{V_0^2 - (\varepsilon_F - \hbar\omega_c(n+1/2))^2}}{\hbar} \tag{27}$$

   where we have reintroduced the sum over Landau bands in order to account for the case where several bands cross the Fermi level (when $\hbar\omega_c < V_0$). It is understood that the contribution is zero when the argument of the square root is negative.

   Assuming that this correction remains small compare to $\sigma_{ij}$ for $V(x)$, we obtain that the longitunial resistivity are

$$\Delta\rho_{xx} \simeq \frac{\Delta\sigma_{yy}}{\sigma_{xx}^2 + \sigma_{xy}^2} \quad \text{and} \quad \Delta\rho_{yy} = 0 \tag{28}$$

   In the experiment of Weiss et al, the parameters are :
   • $\varepsilon_F = 11$ meV
   • $0 < B < 0.8$ T, i.e. $0 < \hbar\omega_c < 1.2$ meV

- $V_0 = 0.3\,\text{meV}$
- $T = 2.2\,\text{K}$ i.e. $k_B T = 0.19\,\text{meV}$
- $\hbar/\tau = 0.013\,\text{meV}$

If we plot (28) with the $T = 0$ expression (27), we obtain the Fig. 3 (the range corresponds to $0 < B < 0.8\,\text{T}$).

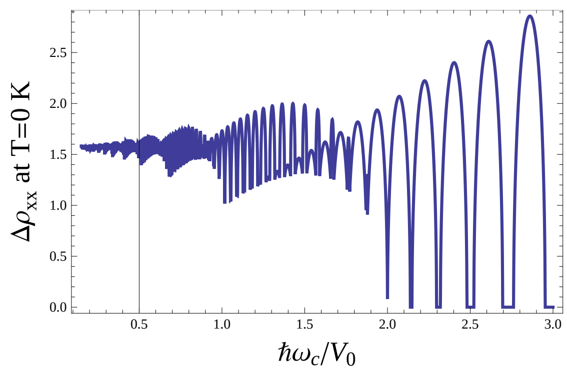



FIGURE 3 – *Weiss oscillations at $T = 0\,\text{K}$.*

For a finite temperature, all the rapid oscillations are smoothed and only the smooth envelope remains.

5. The form of the smooth envelope can be understood by a semiclassical argument : the transport properties are controlled by cyclotron orbits. The electron of energy $\varepsilon_F$ (longitudinal conductivity is a Fermi surface property) has a circular trajectory of radius $R_c = \frac{\sqrt{2\varepsilon_F/m_*}}{\omega_c} = \ell_B^2 k_F$.

   The periodic potential $V(x)$ is equivalent to a periodic electric field $\vec{\mathscr{E}} = -\vec{u}_x V'(x)/e$.

   If the radius is commensurate with the period, $2nR_c = a$, the electron feels an electric field with the same sign at the left and the right of the orbit, what induces a drift in the $y$ direction, hence the anisotropy in the transport properties. The period is

$$\Delta(1/B) \sim \frac{e\,a}{\hbar k_F} \tag{29}$$

   On the experimental data (Fig. 1), we can see that, in the low field part of the graph ($B \lesssim 0.4$ T) the resistivity $\rho_{xx}$ oscillates while $\rho_{yy}$ is flat. These oscillations becomes slower as $B$ increases. This is consistent with our analysis.

6. The rapid oscillations appearing at larger magnetic field ($B \gtrsim 0.4\,\text{T}$) are **Shubnikov-de Haas oscillations**.

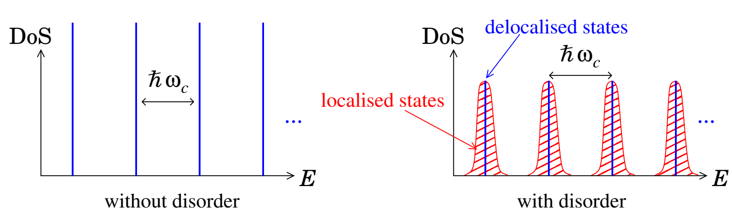



FIGURE 4 – *Landau spectrum (density of states) without and with disorder.*

The expected density of states (DoS) of a disordered 2DEG is represented on Fig. 4. The oscillations of the DoS (i.e. the quantisation of energy) are responsible for oscillations of thermodynamic quantities (like the de Haas-van Alphen effect for the magnetisation).

Concerning the transport properties : the constant relaxation rate approximation ($0^+ \to \hbar/\tau$) has led to the semi-classical (Drude-Sommerfeld) expression for $\rho_{xx} = 1/\sigma_0$ (with no oscillation). A more reallistic treatment of scattering processes on the disordered potential (by self consistent Born approximation for example) shows that the conductivity presents oscillations as a function of $1/B$ similar to oscillations of thermodynamic quantities (remind that the conductivity is proportional to the DoS by the Einstein relation $\sigma_{xx} = e^2 \rho(\varepsilon_F) D$). We emphasize that, contrarily to the case of thermodynamic properties, the conductivity oscillations relies crucially on the presence of disorder : it localises a fraction of eigenstates which do not participate to transport. The period of SdH oscillations is

$$\Delta(1/B) = \frac{2e}{\hbar k_F^2} = \frac{2e}{n_e h} \tag{30}$$

Remark : the study of SdH oscillations is a common experimental tool to determine the density of charge carriers at low temperature.

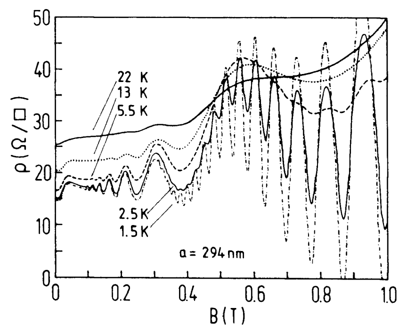



**Figure 4.** $\rho_{xx}$ **vs. B in the temperature range between 1.5 and 22K. The commensurability oscillations are less temperature dependent compared to the SdH-oscillations.**

FIGURE 5 – *From D. Weiss, ≪ Magnetoquantum oscillations in a lateral superlattice ≫ (1990).*

The two types of oscillations (Weiss oscillations and SdH oscillations) do not have the same period (besides, the Weiss oscillations signals anisotropic transport). They have different temperature dependences : the SdH oscillations rapidly disappear due to thermal broadening whereas the Weiss oscillations are weakly dependent on temperature (see Fig. 5).

☞ To learn more :

• Experiment analysed here was reported in : R. R. Gerhardts, D. Weiss and K. von Klitzing, *Novel magnetoresistance oscillations in a periodically modulated two-dimensional electron gas*, Phys. Rev. Lett. **62**, 1173 (1989).

• A semiclassical theory, which is more appropriate to describe the experiment, was developed in : C. W. J. Beenakker, *Guiding-center-drift resonance in a periodically modulated two-dimensional electron gas*, Phys. Rev. Lett. **62**, 2020 (1989) ; R. R. Gerhardts, *Quasiclassical calculation of magnetoresistance oscillations of a 2D electron gas in an anharmonic lateral superlattice potential*, Phys. Rev. B **45**, 3449 (1989).

• A little review which may be found on the internet is : D. Weiss, ≪ Magnetoquantum oscillations in a lateral superlattice ≫, pp. 133–150, in Electronic properties of multilayers and low-dimensional semiconductors structures, Edited by J. M. Chamberlain et al., Plenum Press, New York, 1990.

• The detailed analysis of the Landau problem can by found in : L. Landau & E. Lifchitz, volume 3 or my book, C. Texier, ≪ Mécanique quantique ≫, chapter 16, 2nd edition, Dunod (2015). Discussion of the Feynman-Hellmann theorem can be found in my book as well.