Out of equilibrium statistical physics – Exam

Wednesday 6 january 2016

Write your answers to this part of the exam on a **SEPARATE** sheet.

Vous pouvez rédiger en français si vous le souhaitez.

Subject : **Zero sound in Helium-3**

Introduction : In usual fluids, the propagation of sonic waves (first sound) is explained by compression waves where local thermal equilibrium is ensured. Dissipation arises from temperature and velocity gradients. The absorption coefficient of a wave of frequency ω is found to depend on temperature as $\Gamma_1 \sim \omega^2/T^2$; such increase at low temperature makes propagation of first sound impossible at low T. In 1957, Landau predicted the existence of another type of waves in electrically neutral Fermi liquids, which can propagate at zero temperature, hence the denomination ≪ *zero sound* ≫. These waves are of different nature than first sound as they arise from Fermi distribution deformations (an out-of-equilibrium phenomenon). Dissipation of zero sound is controlled by the collision rate in Fermi liquid, which vanishes at low temperature as $\Gamma_0 \sim T^2$.

Parts A & B are (almost) independent

A. Compressibility of non-interacting Fermions.– We first derive the compressibility of the Fermi gas within the model of non-interacting free fermions. The one-body Hamiltonian is simply

$$\hat{H}_0^{(\text{one body})} = \frac{\hat{\vec{p}}^{\,2}}{2m_*} \tag{1}$$

where m_* is the effective mass. Eigenstates are simply the plane waves $\varphi_{\vec{k}}(\vec{r}) = \frac{1}{\sqrt{V}}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}}$ of energy $\varepsilon_{\vec{k}} = \vec{k}^{\,2}/(2m_*)$, with $\hbar = 1$ (V is the volume ; we recall that the momentum is quantised in finite volume). We introduce the Fourier component of the one body density operator

$$\hat{n}_{\vec{q}} \stackrel{\text{def}}{=} \int_V \mathrm{d}^3\vec{r}\,\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r}}\,\hat{n}(\vec{r}) = \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\hat{\vec{r}}} \tag{2}$$

where $\hat{\vec{r}}$ is the position operator.

1/ Calculate the matrix element $\langle\,\varphi_{\vec{k}}\,|\hat{n}_{\vec{q}}|\,\varphi_{\vec{k}\,'}\,\rangle$.

2/ An external scalar potential is applied, leading to the perturbation Hamiltonian

$$\hat{H}_{\text{ext}}(t) = \int \mathrm{d}^3\vec{r}\;\hat{n}(\vec{r})\,V^{\text{ext}}(\vec{r},t) = \frac{1}{V}\sum_{\vec{q}} \hat{n}_{-\vec{q}}\,V^{\text{ext}}_{\vec{q}}(t)\,. \tag{3}$$

The compressibility is defined by

$$\big\langle \hat{n}_{\vec{q}}(t) \big\rangle_{V^{\text{ext}}} = \big\langle \hat{n}_{\vec{q}} \big\rangle + \int_{-\infty}^{+\infty} \mathrm{d}t'\,\chi_{\vec{q}}(t-t')\,V^{\text{ext}}_{\vec{q}}(t') + \cdots \tag{4}$$

Express the response function as an equilibrium correlation function.

3/ We denote by $\chi_0(\vec{q},\omega)=\int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\,\chi_{\vec{q}}(t)$ the compressibilty of the *fermion gas* where $\chi_{\vec{q}}(t)$ now denotes the response function of the *many body* problem. Express it in terms of the matrix elements of the one body operator of question **1** (Hint : use appendix).

4/ Show that $\lim_{\vec{q}\to0}\lim_{\omega\to0}\chi_0(\vec{q},\omega)$ is related to the density of states ν_0 per unit volume at Fermi energy ε_F, assuming that the temperature is small compared to the Fermi energy, $k_BT\ll\varepsilon_F$. Justify physically the sign.

B. Interacting Fermions.– We first reformulate the result of part **A** by introducing the notation

$$\delta n(\vec{q},\omega)\stackrel{\mathrm{def}}{=}\int\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\,\left(\langle\hat{n}_{\vec{q}}(t)\rangle_{V^{\mathrm{ext}}}-\langle\hat{n}_{\vec{q}}\rangle\right)\tag{5}$$

for the extra density induced by the external perturbation in the gas. $\delta n(\vec{q},\omega)$ is controlled by the compressibilty $\chi_0(\vec{q},\omega)$ studied in part **A** :

$$\delta n(\vec{q},\omega)=\chi_0(\vec{q},\omega)\,V^{\mathrm{ext}}(\vec{q},\omega)\,.\tag{6}$$

We now consider a fluid of **interacting fermions** described by the many-body Hamiltonian

$$\hat{H}=\hat{H}_0+\hat{H}_{\mathrm{int}}=\hat{H}_0+\frac{1}{2}\int\mathrm{d}^3\vec{r}\mathrm{d}^3\vec{r}\,'\,\hat{n}(\vec{r})\,u(\vec{r}-\vec{r}\,')\,\hat{n}(\vec{r}\,')\,.\tag{7}$$

Due to the interaction, a variation of the density produces a contribution to the potential :

$$V^{\mathrm{Hartree}}(\vec{r})=\frac{\delta H_{\mathrm{int}}}{\delta n(\vec{r})}=\int\mathrm{d}^3\vec{r}\,'\,u(\vec{r}-\vec{r}\,')\,n(\vec{r}\,')\,.\tag{8}$$

We now follow a *self consistent* approach in order to determine the compressibility $\chi(\vec{q},\omega)$ for the interacting fermions. Introducing $\tilde{u}(\vec{q})=\int_V\mathrm{d}^3\vec{r}\,\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r}}\,u(\vec{r})$, we write the total potential as

$$V^{\mathrm{tot}}(\vec{q},\omega)=V^{\mathrm{ext}}(\vec{q},\omega)+V^{\mathrm{Hartree}}(\vec{q},\omega)=V^{\mathrm{ext}}(\vec{q},\omega)+\tilde{u}(\vec{q})\,\delta n(\vec{q},\omega)\,.\tag{9}$$

In the self consistent approach, one replaces Eq. (6) of part **A** by the equation

$$\delta n(\vec{q},\omega)=\chi_0(\vec{q},\omega)\,V^{\mathrm{tot}}(\vec{q},\omega)\,.\tag{10}$$

1/ Deduce the expression of the compressibility of the interacting gas, defined by $\delta n(\vec{q},\omega)=\chi(\vec{q},\omega)\,V^{\mathrm{ext}}(\vec{q},\omega)$. Express $\chi(\vec{q},\omega)$ in terms of $\chi_0(\vec{q},\omega)$ and $\tilde{u}(\vec{q})$.

A lengthy calculation shows that the compressibility for the non-interacting fermions found in part **A** is $\chi_0(\vec{q},\omega)\simeq+\nu_0\,\frac{1}{3}\big(\frac{v_Fq}{\omega}\big)^2$ for $v_Fq\ll\omega$, where $q\stackrel{\mathrm{def}}{=}||\vec{q}||$ and v_F is the Fermi velocity.

For **electrically neutral** fermions like Helium-3 atoms, interaction can be considered purely local, i.e $\tilde{u}(\vec{q})\simeq\tilde{u}(0)=\mathrm{cst}$.

2/ Show that the compressibility presents the behaviour

$$\chi(\vec{q},\omega)\propto\frac{q^2}{\omega^2-\omega_0(q)^2}\,.\tag{11}$$

Discuss the physical meaning of such a structure. Identify an important aspect which has been forgotten in this derivation and correct the expression.

3/ We recall the expressions of the density of states $\nu_0=m_*k_F/(\pi\hbar)^2$ and the fermion density $n=k_F^3/(3\pi^2)$. Identify the speed of zero sound c_0 in the dispersion relation and express it as a function of n, m_* and the interaction strength $\tilde{u}(0)$.

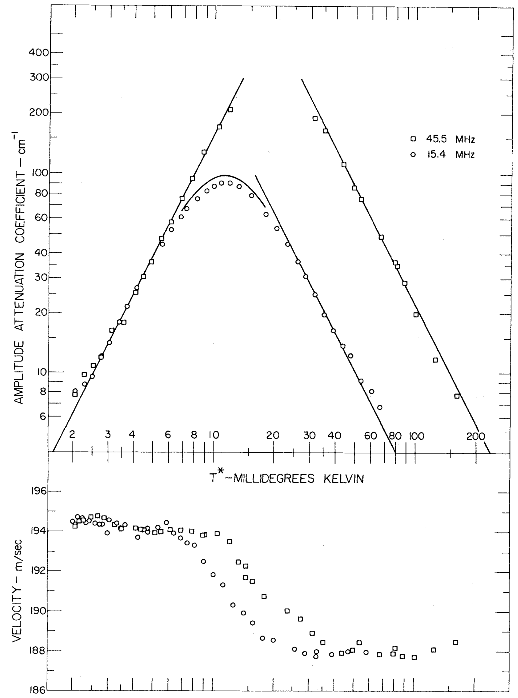

FIGURE 1 : *Dissipation rate and celerity of sound waves in liquid Helium-3 at* $p = 0.32\,\mathrm{atm}$ *as a function of temperature.*
From : W. R. Abel, A. C. Anderson and J. C. Wheatley, ≪ *Propagation of zero sound in He*^{3} *at low temperatures* ≫, Phys. Rev. Lett. **17**, 74 (1966).

4/ Discuss the experimental result of Fig. 1 (at the light of the discussion of the introduction).

5/ A.N. : The Fermi velocity given in the article is $v_F = 54\,\mathrm{m/s}$. Deduce the value of the interaction strength $\tilde{u}(0)$ in meV.nm^3 (we recall that the proton mass is $m_p = 1.67 \times 10^{-27}$ kg).

6/ BONUS **: Comparison with electrically charged Fermi liquid.–** We now compare the previous discussion with the case of *charged* Fermi liquids. In this case the interaction is the Coulomb interaction $\tilde{u}(\vec{q}) = 4\pi e^2/q^2$, where $e^2 = q_e^2/(4\pi\epsilon_0) = 1.44\,\mathrm{eV.nm}$. Given that the two first terms of the expansion of the compressibility of the free fermion gas are

$$\chi_0(\vec{q},\omega) \simeq +\nu_0 \left[\frac{1}{3}\left(\frac{v_F q}{\omega}\right)^2 + \frac{1}{5}\left(\frac{v_F q}{\omega}\right)^4\right] \qquad \text{for } v_F q \ll \omega\,, \tag{12}$$

analyse the compressibility $\chi(\vec{q},\omega)$ in this case (show that it involves the characteristic frequency $\omega_p = \sqrt{4\pi n e^2/m_*}$). Deduce a new dispersion relation $\omega_{\mathrm{pl}}(q)$. Compare with the dispersion relation of the neutral Fermi liquid. Estimate ω_p for $n \simeq 6 \times 10^{28}\ \mathrm{m}^{-3}$ and $m_* \simeq 10^{-30}$ kg (electrons in silver).

☞ Appendix :

We recall that the grand canonical average of a commutator of many body operators, sums of one particle operators, of the form $\hat{A} = \sum_{i=1}^{N} \hat{a}^{(i)}$ and $\hat{B} = \sum_{i=1}^{N} \hat{b}^{(i)}$, is

$$\langle[\hat{A}\,,\hat{B}]\rangle = \sum_{\alpha} f_\alpha \,\langle\, \varphi_\alpha \,|\, [\hat{a},\hat{b}] \,|\, \varphi_\alpha \,\rangle = \sum_{\alpha,\,\beta} (f_\alpha - f_\beta)\, a_{\alpha\beta}\, b_{\beta\alpha}\,, \tag{13}$$

where the sum runs over one particle stationary states. $a_{\alpha\beta} = \langle\, \varphi_\alpha \,|\hat{a}|\, \varphi_\beta \,\rangle$ is a matrix element of the one particle operator and $f_\alpha = f(\varepsilon_\alpha)$ denotes the occupancy of the individual eigenstate.

Solutions à l'adresse `http://www.lptms.u-psud.fr/christophe_texier/`

Zero sound in Helium-3 – Answers

A. Compressibility of non-interacting Fermions.

1/ Preliminary : we need the matrix element

$$\langle \varphi_{\vec k}|\hat n_{\vec q}|\varphi_{\vec k\,'}\rangle = \int_V \frac{\mathrm{d}^3\vec r}{V}\,\mathrm{e}^{-\mathrm{i}\vec k\cdot\vec r}\mathrm{e}^{-\mathrm{i}\vec q\cdot\vec r}\mathrm{e}^{\mathrm{i}\vec k\,'\cdot\vec r} = \delta_{\vec k\,',\vec k+\vec q}$$

2/ Within the notations of the course : $f(t)\to V^{\mathrm{ext}}_{\vec q}(t)$, $A\to -(1/V)\hat n_{-\vec q}$ and $B\to\hat n_{\vec q}$, thus the response function is

$$\chi_{\vec q}(t) = -\frac{\mathrm{i}}{V}\,\theta_{\mathrm{H}}(t)\left\langle\left[\hat n_{\vec q}(t)\,,\,\hat n_{-\vec q}\right]\right\rangle \tag{14}$$

3/ Using the formula of the appendix we get

$$\chi_0(\vec q,\omega) = -\frac{\mathrm{i}}{V}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{(\mathrm{i}\omega-0^+)t}\sum_{\vec k,\vec k\,'}\left(f_{\vec k}-f_{\vec k\,'}\right)\mathrm{e}^{\mathrm{i}\varepsilon_{\vec k}t}\langle\varphi_{\vec k}|\hat n_{\vec q}|\varphi_{\vec k\,'}\rangle\mathrm{e}^{-\mathrm{i}\varepsilon_{\vec k\,'}t}\langle\varphi_{\vec k\,'}|\hat n_{-\vec q}|\varphi_{\vec k}\rangle$$
$$= \frac{1}{V}\sum_{\vec k}\frac{f_{\vec k}-f_{\vec k+\vec q}}{\omega+\varepsilon_{\vec k}-\varepsilon_{\vec k+\vec q}+\mathrm{i}0^+}$$

4/ We first consider the limit $\omega\to 0$, then

$$\chi_0(\vec q,0) = \frac{1}{V}\sum_{\vec k}\frac{f_{\vec k}-f_{\vec k+\vec q}}{\varepsilon_{\vec k}-\varepsilon_{\vec k+\vec q}}$$

(the regulator is not needed as the fraction is regular for $\vec q\to 0$). In the small wavevector limit we can write

$$\lim_{\vec q\to 0}\chi_0(\vec q,0) = \frac{1}{V}\sum_{\vec k}\frac{\partial f_{\vec k}}{\partial\varepsilon_{\vec k}} \simeq -\frac{1}{V}\sum_{\vec k}\delta(\varepsilon_F-\varepsilon_{\vec k}) = -\nu_0$$

(in usual metals we can always assume $k_BT\ll\varepsilon_F$). The negative sign is interpreted as follows : a positive uniform static potential changes the spectrum as $\varepsilon_{\vec k}=\frac{\vec k^{\,2}}{2m_*}\longrightarrow\varepsilon_{\vec k}=\frac{\vec k^{\,2}}{2m_*}+V^{\mathrm{ext}}$. At fixed ε_F, the electron density is thus diminished by $\delta n\simeq -\nu_0\,V^{\mathrm{ext}}$.

Remark : Note that the two limits do not commute. We see from (12) that $\lim_{\vec q\to 0}\chi_0(\vec q,\omega)=0$.

B. Interacting Fermions.

1/ We use the self consistent argument : we replace $\delta n(\vec q,\omega)=\chi_0(\vec q,\omega)\,V^{\mathrm{ext}}(\vec q,\omega)$ for the non-interacting fermions by $\delta n(\vec q,\omega)=\chi_0(\vec q,\omega)\,V^{\mathrm{tot}}(\vec q,\omega)$, thus

$$\delta n(\vec q,\omega) = \chi_0(\vec q,\omega)\,\left[V^{\mathrm{ext}}(\vec q,\omega)+\tilde u(\vec q)\,\delta n(\vec q,\omega)\right]$$

leading to

$$\delta n(\vec q,\omega) = \underbrace{\frac{\chi_0(\vec q,\omega)}{1-\chi_0(\vec q,\omega)\tilde u(\vec q)}}_{=\chi(\vec q,\omega)}\,V^{\mathrm{ext}}(\vec q,\omega)$$

This coincides with the RPA approximation

$$\boxed{\chi(\vec{q},\omega) = \frac{1}{1/\chi_0(\vec{q},\omega) - \tilde{u}(\vec{q})}} \tag{15}$$

2/ Assuming constant interaction and using the expansion of the free compressibility we get

$$\chi(\vec{q},\omega) \simeq \frac{1}{\frac{3}{\nu_0}\left(\frac{\omega}{v_F q}\right)^2 - \tilde{u}(0)} = \frac{\nu_0 v_F^2}{3}\,\frac{q^2}{\omega^2 - (\nu_0/3)v_F^2\tilde{u}(0)\,q^2}$$

Defining $c_0 \stackrel{\text{def}}{=} v_F\sqrt{\nu_0\tilde{u}(0)/3}$ and re-introducing the regulator ensuring causality we obtain

$$\boxed{\chi(\vec{q},\omega) \simeq \frac{\nu_0}{3}\,\frac{(v_F q)^2}{(\omega + \mathrm{i}0^+)^2 - c_0^2\,q^2}} \tag{16}$$

The resonance line in the place (q,ω) corresponds to the spectrum of *collective excitations* with a *linear* dispersion relation $\omega = \omega_0(q) = c_0 q$ (like usual sound). Since we are studying density modulations, these collective excitations are density waves mediated by the interaction : this is the zero sound.

3/ Using the expression of the DoS and the density we can rewrite the zero sound velocity as

$$c_0 = \sqrt{\frac{n\,\tilde{u}(0)}{m_*}} = v_F\,\sqrt{\frac{\nu_0\tilde{u}(0)}{3}}\;. \tag{17}$$

The measure of sound velocity directly provides the interaction strength. Nice!

4/ The experimental data shows a clear crossover between the dissipation rate $\Gamma_1 \sim \omega^2/T^2$ characteristic for first sound (high T) to the collision rate $\Gamma_0 \sim T^2$ (low T) between fermions controlling the damping of zero sound. This is a clear indication that sound waves characterised by celerity $c_0 \simeq 194$ m/s (for $T \lesssim 10$ mK) is zero sound.

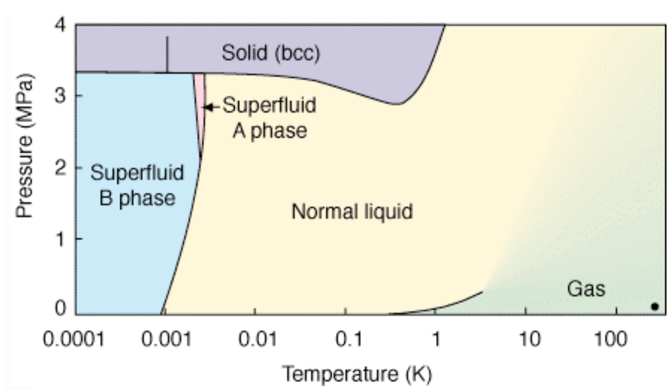

FIGURE 2 : *Phase diagram of Helium-3. Experiment was realised at* $p = 0.32$ atm.

5/ A.N. : Given the Fermi velocity $v_F = 54$ m/s (measured by Wheatley) we deduce the Fermi energy $\varepsilon_F = 45\ \mu$eV ($T_F = 0.53$ K), the Fermi wavevector $k_F = 2.7\ \text{nm}^{-1}$ and the density $n = k_F^3/(3\pi^2) = 0.66\ \text{nm}^{-3}$ (this corresponds to a volumic mass $\rho = 3.3$ g/liter).[1]

[1] Temperature and preassure dependences of the density of Helium-3 were studied in the article : J. E. Rives and H. Meyer, Phys. Rev. Lett. **7**, 217 (1961).
Note that Helium-3 boils at $T = 3.19$ K (at $p = 1$ atm). At the boiling point, the mass per unit volume is $\rho = 59$ g/liter (Wikipedia).

The density of states is $\nu_0 = 3n/(2\varepsilon_F) = 22\,\text{meV}^{-1}.\text{nm}^{-3}$.
In the experiment, Abel *et alter* measured the zero sound velocity $c_0 \simeq 194$m/s, thus $\tilde{u}(0)\nu_0 \simeq 39$ i.e. $\tilde{u}(0) \simeq 1.76\,\text{meV.nm}^3$.

Remark : The form $\chi_0 \simeq \frac{1}{3}\nu_0\left(\frac{v_F q}{\omega}\right)^2$ is only the first term of the expansion (12). If instead of one term, one has kept the two first terms, the denomitor of χ would take the form

$$\omega^2\left[1-\chi_0(\vec{q},\omega)\,\tilde{u}(0)\right] = \frac{1}{\omega^2}(\omega^2-\lambda_+)(\omega^2-\lambda_-)$$

with $\lambda_\pm = (c_0 q)^2\frac{1}{2}\left[1\pm\sqrt{1+36/(5\tilde{u}(0)\nu_0)}\right]$. For $\tilde{u}(0)\nu_0 \gg 1$, the denomitor then reads

$$\omega^2\left[1-\chi_0(\vec{q},\omega)\,\tilde{u}(0)\right] \simeq (\omega^2-(c_0q)^2)\frac{\omega^2+9(c_0q)^2/(5\tilde{u}(0)\nu_0)}{\omega^2} \simeq \omega^2-(c_0q)^2\;,$$

for small q and finite ω. The calculation performed above is thus valid in the limit $\tilde{u}(0)\nu_0 \gg 1$.

6/ Electrically charged Fermi liquids.– If we introduce the Coulomb interaction in the RPA compressibility, we obtain a completly different result,

$$\chi(\vec{q},\omega) = \frac{\chi_0(\vec{q},\omega)}{1-\chi_0(\vec{q},\omega)4\pi e^2/q^2} = \frac{\nu_0}{3}\frac{(v_Fq)^2+\cdots}{\omega^2-4\pi e^2\nu_0 v_F^2\left[\frac{1}{3}+\frac{1}{5}\left(\frac{v_Fq}{\omega}\right)^2+\cdots\right]}\;, \tag{18}$$

as the q^2 in χ_0 simplifies with the $1/q^2$ in $\tilde{u}$. As for neutral fermions, we must identify the pole.

a) Shortcut : for $q\to 0$ the denominator is simply $\omega^2-\omega_p^2$ where we have introduced the *plasma* frequency $\omega_p \stackrel{\text{def}}{=} v_F\sqrt{4\pi e^2\nu_0/3} = \sqrt{4\pi n e^2/m_*}$. For finite q we may therefore simply replace the $1/\omega^2$ by $1/\omega_p^2$ in the denominator of (18). Thus, reintroducing the regulator ensuring causality, we deduce

$$\boxed{\chi(\vec{q},\omega) \simeq \frac{\nu_0}{3}\frac{(v_Fq)^2}{(\omega+\mathrm{i}0^+)^2-\omega_p^2-(3/5)(v_Fq)^2}}$$

We deduce the dispersion relation for the plasmon modes

$$\omega_{\text{pl}}(q) = \sqrt{\omega_p^2+c_p^2q^2} \qquad \text{with} \qquad c_p = v_F\sqrt{\frac{3}{5}}\;,$$

characterising *charge density waves*. For silver we obtain $\omega_p \simeq 1.3\times10^{16}\,\text{s}^{-1} = 8.1\,\text{eV}$, i.e. a very big gap.

b) Direct analysis : If you prefer, you can write

$$\omega^2-4\pi e^2\nu_0 v_F^2\left[\frac{1}{3}+\frac{1}{5}\left(\frac{v_Fq}{\omega}\right)^2\right] = \frac{1}{\omega^2}(\omega^2-\lambda_+)(\omega^2-\lambda_-)$$

with now $\lambda_\pm = \omega_p^2\frac{1}{2}\left[1\pm\sqrt{1+(12/5)(v_Fq/\omega_p)^2}\right]$. Hence the denominator of (18) is

$$\simeq\left(\omega^2-\omega_p^2-\frac{3}{5}(v_Fq)^2\right)\frac{\omega^2+(3/5)(v_Fq)^2}{\omega^2} \simeq \omega^2-\omega_p^2-(3/5)(v_Fq)^2$$

for small q and finite ω.