TD n°4 : Out of equilibrium statistical physics
Classical formulations (1) : Langevin approach

1 Gaussian *versus* non Gaussian white noise – Shottky noise

We consider the noise

$$F(t) = \sum_{n=1}^{N} \kappa_n \, \delta(t - t_n) \qquad \text{for } t \in [0, T] \tag{1}$$

where N is random. $\{\kappa_n\}$ and $\{t_n\}$ are two sets of i.i.d. random variables. [1] The probability to have N "impulses" in $[0, T]$ is

$$P_T(N) = \frac{(\lambda T)^N}{N!} \mathrm{e}^{-\lambda T} \tag{2}$$

The t_n are uniformly distributed over the interval $[0, T]$, i.e. the joint distribution of the N times simply $P_N(t_1, \cdots, t_N) = 1/T^N$. The weights κ_n's have a common law $p(\kappa)$.

We first consider the case where $p(\kappa) = \delta(\kappa - q)$.

1/ We introduce the generating functional

$$G[\phi(t)] \stackrel{\text{def}}{=} \left\langle \mathrm{e}^{\int \mathrm{d}t \, \phi(t) \, F(t)} \right\rangle \tag{3}$$

Show how one can deduce the correlation functions from the knowledge of $G[\phi]$ (which will be calculated below).

Hint : Use the functional derivatives $\frac{\delta G}{\delta \phi(t_1)}$, $\frac{\delta^2 G}{\delta \phi(t_1) \delta \phi(t_2)}$, etc. Functional derivatives are easily computed with the rule

$$\frac{\delta \phi(t')}{\delta \phi(t)} = \delta(t - t') \tag{4}$$

and usual rules for derivation. Example : $\frac{\delta}{\delta \phi(t)} \int \mathrm{d}t' \, \phi(t')^2 = 2 \, \phi(t)$.

2/ Using that averaging over the random variables is

$$\langle (\cdots) \rangle_{N, \{t_n\}} = \sum_{N=0}^{\infty} \frac{(\lambda T)^N}{N!} \mathrm{e}^{-\lambda T} \int_0^T \frac{\mathrm{d}t_1}{T} \cdots \frac{\mathrm{d}t_N}{T} (\cdots) \tag{5}$$

compute explicitly $G[\phi(t)]$.

3/ Functional derivations of $G[\phi]$ generate the correlation functions $\langle F(t_1) \cdots F(t_n) \rangle$ and the derivations of $W[\phi] = \ln G[\phi]$ generate the connex correlation functions, i.e. $\langle F(t) \rangle$, $\langle F(t) F(t') \rangle_c \stackrel{\text{def}}{=} \langle F(t) F(t)' \rangle - \langle F(t) \rangle \langle F(t)' \rangle$, etc. Deduce these latter.

4/ Application : Classical theory of shot noise (Shottky noise).– Some current $i(t)$ flows through a conductor. Due to the discrete nature of the charge carriers, the current presents some fluctuations (noise) known as « shot noise », which we aim to characterize here (not to be confused with the thermal fluctuations, i.e. the Johnson-Nyquist noise). We assume that

[1] i.i.d. = independent and identically distributed.

the current can be written under the form of independent implulses $i(t) = q\sum_n \delta(t-t_n)$. The average rate is λ. Express the two first cumulants of current. Deduce the power spectrum

$$S(\omega) \stackrel{\text{def}}{=} \int \mathrm{d}(t-t')\, \mathrm{e}^{\mathrm{i}\omega(t-t')} \langle i(t)\, i(t')\rangle_c \tag{6}$$

and express the relation between the shot noise and the averaged current $\langle i\rangle$.

Remark : This result has permitted to demonstrate the existence of charge carriers with *fractional charge* in the regime of the fractional quantum Hall effect (strong magnetic field, low temperature) :

- L. Saminadayar, D. C. Glattli, Y. Jin & B. Etienne, *Observation of the $e/3$ Fractionally Charged Laughlin Quasiparticle*, Phys. Rev. Lett. **79** (1997) 2526.
- M. Reznikov, R. de Picciotto, T. G. Griffiths, M. Heiblum & V. Umansky, *Observation of quasiparticles with $1/5$ of an electron's charge*, Nature **399** (May 1999) 238.

5/ Transfered charge (Poisson process).– We consider the stochastic differential equation

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = i(t) \tag{7}$$

a) Draw a typical realisation of the process $Q(t)$. Deduce the cumulants of the charge $\langle Q(t)^n\rangle_c$.

b) Argue that on the large time scale $\lambda t \gg 1$, the cumulants with $n > 2$ can be neglected. What is then the nature of the process $Q(t)$?

c) We introduce the distribution of the charge $P(Q;t) = \langle \delta(Q - Q(t))\rangle$ describing the evolution of the process with a drift

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \mathcal{I}(Q(t)) + i(t)\ . \tag{8}$$

Consider separatly the effect of the drift and the jumps to relate $P(Q;t+\mathrm{d}t)$ to $P(Q;t)$. Show that the distribution obeys

$$\partial_t P(Q;t) = -\partial_Q \left[\mathcal{I}(Q)\, P(Q;t)\right] + \lambda \left[P(Q-q;t) - P(Q;t)\right]\ . \tag{9}$$

6/ Compound Poisson process.– We now consider an arbitrary distribution $w(\kappa)$ and introduce the generating function $g(k) = \left\langle \mathrm{e}^{k\kappa_n}\right\rangle$.

a) Find the new expression of the generating functional $G[\phi]$.

b) Show that it is possible to define a limit (changing λ and $w(\kappa)$) where the noise becomes a Gaussian white noise.

c) Show that the generalisation of (9) is

$$\boxed{\partial_t P(Q;t) = -\partial_Q \left[\mathcal{I}(Q)\, P(Q;t)\right] + \lambda \int \mathrm{d}q\, w(q)\ \left[P(Q-q;t) - P(Q;t)\right]} \tag{10}$$

Check the conservation of probability. Express the probability current $\mathcal{J}(Q;t)$ related to the distribution by the conservation law $\partial_t P(Q;t) = -\partial_Q \mathcal{J}(Q;t)$.

Consider the limit of small jumps $q \to 0$, i.e. when $w(q)$ is concentrated at the origin. Assuming $\langle q\rangle = 0$, show that (10) leads to the Fokker-Planck equation and express the diffusion constant D of the charge diffusion.

2 Generalised Langevin equation – Wiener-Khintchine theorem

We consider a small particle in a fluid whose velocity can be analysed thanks to the generalised Langevin equation

$$m\frac{\mathrm{d}}{\mathrm{d}t} v(t) = -\int \mathrm{d}t'\, \gamma(t-t')\, v(t') + F(t) \tag{11}$$

(set $m = 1$). The Langevin force $F(t)$ is correlated over a short "microscopic" time τ_c. The integral term comes from damping : damping sets over a finite memory time $\tau_m \gg \tau_c$, hence $\gamma(\tau)$ is a causal function decaying fast over this time scale, like $\gamma(\tau) = \theta_{\mathrm{H}}(\tau)\,(\lambda/\tau_m)\,\mathrm{e}^{-\tau/\tau_m}$.

1/ Show that the correlation function of the velocity is

$$C_{vv}(\tau) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\widetilde{C}_{FF}(\omega)}{|\tilde{\gamma}(\omega) - \mathrm{i}\omega|^2} \mathrm{e}^{-\mathrm{i}\omega\tau} \tag{12}$$

2/ We consider $\gamma(t) = \lambda\,\delta(t)$ and $C_{FF}(\tau) = \sigma\,\delta(\tau)$. Compute the correlator $C_{vv}(\tau)$ and express σ in terms of the diffusion constant $D \overset{\text{def}}{=} \lim_{t\to\infty} \frac{1}{2t} \left\langle x(t)^2 \right\rangle$.

3/ consider now $C_{FF}(t) = 2D\lambda^2 \frac{1}{2\tau_c} \mathrm{e}^{-|t|/\tau_c}$ with $\tau_c \ll 1/\lambda$. Show that

$$\int_{\mathbb{R}} \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{(\omega^2 + a^2)(\omega^2 + b^2)} = \frac{1}{2(b^2 - a^2)} \left(\frac{1}{a} \mathrm{e}^{-a|t|} - \frac{1}{b} \mathrm{e}^{-b|t|} \right) \tag{13}$$

and deduce $C_{vv}(t)$. Analyze its limiting behaviours.

Discuss the hypothesis $\gamma(t) = \lambda\,\delta(t)$ in this case. For a finite τ_m, give a physical argument to express the correct hierarchy of times $1/\lambda$, τ_m and τ_c.

3 Response function for the Ornstein-Uhlenbeck process

We consider a small ball bound to a substrated by a polymer and submitted to a time dependent external force $f_{\mathrm{ext}}(t)$. The position of the particle is described by the Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\lambda\, x(t) + f_{\mathrm{ext}}(t) + F(t) \tag{14}$$

where $F(t)$ is the Langevin force. We choose to model the force as a Gaussian white noise, $\langle F(t)F(t')\rangle = 2D\,\delta(t - t')$.

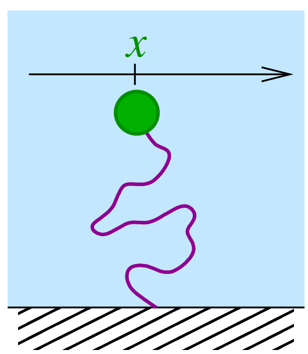

FIGURE 1 : *A small particle is bound to a surface thanks to a polymer which acts like a spring.*

1/ Correlations at equilibrium.– We consider the case $f_{\mathrm{ext}}(t) = 0$. Compute de correlation function $C(t - t') \overset{\text{def}}{=} \langle x(t)x(t')\rangle_{\mathrm{eq}}$. Deduce what is the stationary distribution $P_{\mathrm{eq}}(x)$ of the process. Assuming equipartition theorem, relate D to the temperature.

2/ Response (out of equilibrium).– Show that we can easily determine the response function χ for this linear problem. We recall that it is defined by

$$\langle x(t)\rangle_{\text{out of eq.}} = \langle x\rangle_{\mathrm{eq}} + \int \mathrm{d}t'\, \chi(t - t')\, f_{\mathrm{ext}}(t') + \mathcal{O}(f_{\mathrm{ext}}^2) \tag{15}$$

3/ Fluctuation dissipation theorem.– Check that the two functions satisfy the FDT

$$\boxed{\chi(t) = -\beta\ \theta_{\mathrm{H}}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}C(t)} \tag{16}$$

where $\beta = 1/(k_B T)$.