TD n°5 : Out of equilibrium statistical physics
Classical formulations (2) : Fokker-Planck approach

1 Ornstein-Uhlenbeck process and the Fokker-Planck equation

We study the Ornstein-Uhlenbeck process, the only Markovian, stationnary and Gaussian random process. It describes the motion of the particle submitted to a spring force in the **overdamped** regime. It obeys the Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\lambda\, x(t) + F(t) \tag{1}$$

where $F(t)$ is the Langevin force, a Gaussian white noise $\langle F(t)F(t')\rangle = 2D\,\delta(t-t')$. Our aim is here to determine the stationnary distribution $P_{\mathrm{eq}}(x)$ and the conditional probability $P_\tau(x|x_0)$.

1/ Write the corresponding Fokker-Planck equation.

2/ Method 1.– Recall the expression of $\langle x(t)\rangle$ and $\mathrm{Var}(x(t))$ obtained with the Langevin approach. Deduce the expression of the conditional probability $P_\tau(x|x_0)$. What is its $\tau\to\infty$ limit ?

3/ Method 2.– We solve directly the Fokker-Planck equation. Express the corresponding supersymmetric Schrödinger operator H_+. Give its spectrum of eigenvalues and eigenfunctions. We recall the expression of the quantum mechanical propagator for the harmonic oscillator

$$\langle\, x\,|\mathrm{e}^{-tH_\omega}|\, x_0\,\rangle = \sqrt{\frac{m\omega}{2\pi\,\mathrm{sh}(\omega t)}}\;\exp-\frac{m\omega}{2\,\mathrm{sh}(\omega t)}\left[\mathrm{ch}(\omega t)\,(x^2+x_0^2) - 2\,x\,x_0\right] \tag{2}$$

where $H_\omega = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2x^2$. Recover the expression of the propagator of the Ornstein-Uhlenbeck process.

Appendix : FPE *versus* Schrödinger Equation.– The Fokker-Planck equation

$$\partial_t P(x,t) = \partial_x\big[D\partial_x - F(x)\big]P(x,t) \tag{3}$$

can be mapped onto the imaginary time Schrödinger equation

$$-\partial_t\psi(x,t) = H_+\psi(x,t) \qquad \text{where } H_+ = -D\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{4D}F(x)^2 + \frac{1}{2}F'(x) \tag{4}$$

thanks to the transformation $P(x,t) = \psi_0(x)\,\psi(x,t)$ where $\psi_0(x) = \sqrt{P_{\mathrm{eq}}(x)} \propto \exp[-U(x)/D]$ where $U(x) = -\int^x \mathrm{d}\xi\, F(\xi)$ is the potential.

2 Diffusion in a ring

A. Free diffusion

We consider the free diffusion in a ring

$$\partial_t P(x,t) = D\partial_x^2 P(x,t) \qquad \text{for } x \in [0, L] \tag{5}$$

with periodic boundary conditions

$$P(0) = P(L) \tag{6}$$
$$P'(0) = P'(L) \tag{7}$$

(time dependence is omitted).

1/ Analyze the spectrum of the diffusion operator $D\partial_x^2$. Deduce a first series representation of the propagator $P(x,t|x_0,0)$. Is it convenient to analyze short or large time ? Identify the characterisitic time τ_D (Thouless time) separating the "short" and "long" time regimes.

2/ Using the Poisson formula (appendix), deduce another series representation for $P(x,t|x_0,0)$ convenient to analyze the other limit in time.

B. Effect of a drift

Same question when a constant drift is introduced :

$$\partial_t P(x,t) = \left(D\partial_x^2 - v\,\partial_x\right) P(x,t) \qquad \text{for } x \in [0, L] \tag{8}$$

In particular, discuss the stationary limit $t \to \infty$. Compute the stationary current J_v.

C. Boundary conditions induced current

We now come back to the analysis of the free diffusion (5), however we now study the problem for a new set of boundary conditions :

$$P(L) = 0 \tag{9}$$
$$P'(0) = P'(L) \tag{10}$$

Interpret the two boundary conditions. Found the stationary state and deduce a formula for the current J_D. Discuss the L dependence (compare with J_v).

Remark : the spectral analysis is more tricky in this case because the Fokker-Planck operator in not self adjoint (due to the choice of boundary conditions), which makes it non diagonalisable. The eigenvalues are doubly degenerated and in each subspace the operator must be written under the form of an upper triangular 2×2 matrix.

Appendix : a Poisson formula

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi n\eta}\, \mathrm{e}^{-\pi^2(n+\alpha)^2 y} = \frac{1}{\sqrt{\pi y}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi(n-\eta)\alpha}\, \mathrm{e}^{-\frac{(n-\eta)^2}{y}}\ . \tag{11}$$

Proof : apply $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(2\pi n)$ where $\hat{f}(k) = \int_{\mathbb{R}} \mathrm{d}x\, f(x)\, \mathrm{e}^{-\mathrm{i}kx}$.

3 Few properties of the free diffusion on the line

We illustrate how powerful is the Fokker-Planck approach by considering several properties of the Brownian motion.

1/ Propagator on the half line.– We consider the free diffusion on $\mathbb{R}_+$ with a Dirichlet boundary condition at the origin. Construct the solution of the diffusion equation

$$\partial_t P(x,t) = D\partial_x^2 P(x,t) \qquad \text{for } x > 0 \text{ with } P(0,t) = 0 \tag{12}$$

(use the image method). Apply the method to the propagator, denoted $\mathcal{P}_t^+(x|x_0)$.

2/ Survival probability.– Dirichlet boundary condition describes absorption at $x = 0$. Compute the survival probability for a particle starting from x_0 :

$$S_{x_0}(t) = \int_0^\infty \mathrm{d}x\, \mathcal{P}_\tau^+(x|x_0) \tag{13}$$

Remark : what would have been the result if $\mathcal{P}_\tau^+(x|x_0)$ would have satisfied a Neunmann boundary condition ?

3/ First passage time.– We denote by T the first time at which the process starting from $x_0 > 0$ reaches $x = 0$ (it is a random quantity depending on the process), and $P_{x_0}(T)$ is distribution. The survival probability is the probability that the process did not reach $x = 0$ up to time t :

$$S_{x_0}(t) = \int_t^\infty \mathrm{d}T\, P_{x_0}(T) \tag{14}$$

Deduce $P_{x_0}(T)$ and plot it.

4/ Maximum.– We now consider another property of the Brownian motion $x(\tau)$ with $\tau \in [0,t]$ starting from $x_0 = 0$: we denote by $m \geqslant 0$ its maximum and $Q_t(m)$ the corresponding distribution. Justify the following identity

$$\int_0^m \mathrm{d}m'\, Q_t(m') = S_m(t) \tag{15}$$

Deduce the expression of $Q_t(m)$. What does $Q_t(0)$ represent ? The exponent of the power law $t^{-\theta}$ is called the persistence exponent. Give θ for the Brownian motion.

Appendix : the error function

$$\mathrm{erf}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{d}t\, \mathrm{e}^{-t^2} \tag{16}$$

and $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$. Asymptotics :

$$\mathrm{erfc}(z) \underset{z\to\infty}{\simeq} \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}} \sum_{n=0}^{N} (-1)^n \left(\frac{1}{2}\right)_n \frac{1}{z^{2n+1}} + R_N(z) \tag{17}$$

where $(a)_n \stackrel{\text{def}}{=} a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.