TD n^{o}6 : Out of equilibrium statistical physics Kramers-Kronig relation and causality

1 Kramers-Kronig relations

We denote by $\chi(\omega)$ the Fourier transform of a causal function, assumed summable $\int_{-\infty}^{+\infty}\mathrm{d}\omega\,|\chi(\omega)|<\infty$. We recall the Kramers-Kronig relations :

$$\mathrm{Re}\,\chi(\omega) = \frac{1}{\pi}⨍_{-\infty}^{+\infty}\mathrm{d}\omega'\,\frac{\mathrm{Im}\,\chi(\omega')}{\omega'-\omega} \tag{1}$$

$$\mathrm{Im}\,\chi(\omega) = -\frac{1}{\pi}⨍_{-\infty}^{+\infty}\mathrm{d}\omega'\,\frac{\mathrm{Re}\,\chi(\omega')}{\omega'-\omega} \tag{2}$$

Where the Cauchy principle value is defined by

$$⨍_{-A}^{+B}\mathrm{d}x\,\frac{f(x)}{x} \equiv \int_{-A}^{+B}\mathrm{d}x\,f(x)\,\mathcal{P}\frac{1}{x} = \lim_{\epsilon\to0^+}\left(\int_{-A}^{-\epsilon}+\int_{+\epsilon}^{+B}\right)\mathrm{d}x\,\frac{f(x)}{x} \tag{3}$$

where A, $B>0$. Compute the integral for $f(x)=1$.

1/ Given $\mathrm{Im}\,\chi(\omega)=-\frac{1}{1+\omega^2}$. Deduce $\mathrm{Re}\,\chi(\omega)$ and $\chi(\omega)$.

Hint : In order to compute the Hilbert transform of $\mathrm{Im}\,\chi(\omega)$, integrate $f(z)=\frac{1}{(z-\omega)(1+z^2)}$ over an appropriate contour in the complex plane.

2/ Same question for $\mathrm{Im}\,\chi(\omega)=\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$.

2 Response function in a deterministic case : harmonic and anharmonic oscillators

Reminder : We denote by $B(t)$ a physical observable (ex : the position of a particle). We introduce an external time dependent "force" $f(t)$ coupled to another observable $A(t)$, i.e. energy is $H_{\mathrm{ext}}(t)=-f(t)A$. The evolution of the observable B can be linearised as

$$B_f(t) = B_{f=0}(t) + \int_{-\infty}^{+\infty}\mathrm{d}t'\,\chi_{BA}(t-t')f(t') + O(f^2) \tag{4}$$

where $\chi_{BA}(t)$ is the response function.

1/ Harmonic oscillator.– We consider the harmonic oscillator described by the equation of motion $\ddot{x}+\omega_0^2x=\frac{1}{m}f(t)$. Show that the response function $\chi(t)$ characterising the response of $x(t)$ to the force $f(t)$ is the Green's function of the differential equation. Check that the causal Green's function is $\chi(t)=\theta_{\mathrm{H}}(t)\frac{\sin\omega_0t}{m\omega_0}$. Compute its Fourier transform $\widetilde{\chi}(\omega)$ (for this purpose it is necessary to introduce a regulator $\mathrm{e}^{-\epsilon t}$ with $\epsilon\to0^+$ in the integral). Plot neatly $\widetilde{\chi}(\omega)$.

2/ Damped harmonic oscillator.– We consider now a damped harmonic oscillator submitted to the external force :

$$\ddot{x}+\frac{2}{\tau}\dot{x}+\omega_0^2x=\frac{1}{m}f(t) \tag{5}$$

Compute the Fourier transform of the response function $\widetilde{\chi}(\omega)$. Analyse the poles of this function (for the various regimes). Interpret their positions. Plot neatly $\mathrm{Re}\,\widetilde{\chi}(\omega)$ and $\mathrm{Im}\,\widetilde{\chi}(\omega)$ in the weak damping limit (to be defined). Come back to the first question and interpret physically the regulator $\epsilon \to 0^+$.

3/ Anharmonic oscillator.– We now consider the classical anharmonic oscillator described by the equation of motion $\ddot{x} = F(x)$, where $F(x)$ derives from a confining potential (e.g. $V(x) = \frac{1}{2}\omega^2 x^2 + \frac{1}{4}\lambda x^4$). Deduce the differential equation satisfied by the response function characterizing the out-of-equilibrium situation $\ddot{x} - F(x) = f(t)$. Discuss the differences with the harmonic case.