TD n°7 : Detailed balance and thermalization

1 Two level system coupled to a macroscopic system

Consider a two level system (e.g. : an atom, a spin, a Cooper pair box, etc) coupled to a macroscopic system supposed at *thermal equilibrium*. The Hamiltonian of the whole system is :

$$H = \underbrace{-\frac{\omega_0}{2}\,\sigma_z}_{H_{\text{micro}}} + H_{\text{int}} + H_{\text{macro}} = H_0 + H_{\text{int}} \tag{1}$$

where the Pauli matrix acts in the base $\{|\,g\,\rangle, |\,e\,\rangle\}$. We denote by $\{|\,\Phi_n\,\rangle\}$ a basis of eigenstates for H_{macro}. Interaction Hamiltonian between microscopic and macroscopic is chosen under the form

$$H_{\text{int}} = -\sigma_x\, X \tag{2}$$

where X is an observable of the macroscopic system.

1/ Determine the time evolution of the operator $\sigma_x(t) = \mathrm{e}^{\mathrm{i}H_{\text{micro}}t}\sigma_x\mathrm{e}^{-\mathrm{i}H_{\text{micro}}t}$ (i.e. in the interaction representation). Deduce the matrix element $\langle\, e\,|\sigma_x(t)|\, g\,\rangle$.

Hint : introduce $\sigma_\pm \stackrel{\text{def}}{=} \sigma_x \pm \mathrm{i}\sigma_y$.

2/ We remind the expression of the time evolution operator in the interaction picture

$$\mathcal{U}(t) = T_t \exp\left\{-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau\, V_I(\tau)\right\} \tag{3}$$

where T_t is the time ordering operator. The perturbation in this picture is $V_I(t) = \mathrm{e}^{\mathrm{i}H_0 t/\hbar}H_{\text{int}}\mathrm{e}^{-\mathrm{i}H_0 t/\hbar}$ where $H_0 = H_{\text{micro}} + H_{\text{macro}}$. We choose the initial condition $|\,\psi(0)\,\rangle = |\,g\,\rangle \otimes |\,\Phi_n\,\rangle$. Compute the transition probability between the two energy levels $|\,g\,\rangle \to |\,e\,\rangle$

$$\mathcal{P}_{\text{abs}}^{(\Phi_n)}(t) \stackrel{\text{def}}{=} \sum_m \left|\big(\langle\, e\,| \otimes \langle\, \Phi_m\,|\big)|\,\psi(t)\,\rangle\right|^2 , \tag{4}$$

at lowest order in the perturbation.

3/ The macroscopic is assumed at thermal equilibrium. Show that the absorption rate is

$$\Gamma_{\text{abs}} = \sum_n P_n \frac{\mathrm{d}\mathcal{P}_{\text{abs}}^{(\Phi_n)}(t)}{\mathrm{d}t} \tag{5}$$

where P_n is the probability of occupation of eigenstate $|\,\Phi_n\,\rangle$. For example $P_n \propto \mathrm{e}^{-\beta E_n}$. Show that the absorption rate is controlled by the *unsymmetrised* correlation function $C_{XX}(t) = \langle X(t)X(0)\rangle$.

4/ Get a similar relation for the emission rate Γ_{em} (transition rate for $|\,e\,\rangle \to |\,g\,\rangle$). Use these results to shed light on the detailed balance relation

$$\widetilde{C}_{XX}(-\omega) = \widetilde{C}_{XX}(\omega)\,\mathrm{e}^{-\beta\hbar\omega} . \tag{6}$$

Remark : This discussion has its inspiration in a lecture of Benoît Douçot and in the paper : R. J. Schoelkopf, A. A. Clerk, S. M. Girvin, K. W. Lehnert and M. H. Devoret, *Qubits as spectrometers of quantum noise*, contribution to "Quantum Noise" (Yu. V. Nazarov and Ya. M. Blanter, eds.) (2002), preprint cond-mat/0210247.