# TD n$^o$9 : Structure of the magnetic susceptibility of a ferromagnet - spin waves and magnons

We study a phenomenological quantum model of ferromagnet. The spin density operator $\hat{\vec{S}}(\vec{r})$ (local magnetization) describes magnetic moments localised on atoms or spin of conduction electrons. We consider the following Hamiltonian :

$$\hat{H}_0 = \frac{1}{2}\int \mathrm{d}\vec{r}\,\mathrm{d}\vec{r}\,'\, v(\vec{r}-\vec{r}\,')\,\hat{\vec{S}}(\vec{r})\cdot\hat{\vec{S}}(\vec{r}\,')\,, \tag{1}$$

where $v(\vec{r}-\vec{r}\,')$ is some (decaying) interaction. We recall that magnetization of usual materials is due to an *effective* interaction whose origin lies in the Coulomb interaction and the Pauli principle (exchange or super-exchange mechanisms). Our aim is to analyze the magnetic susceptibility of this model, assuming that the system is in the **ferromagnetic phase**.

*Remark :* When necessary, the operator nature of the observables is underlined with a hat.

**Method :** We follow the "equation of motion theory" :[1] the idea is to develop approximations based on the equations of motion for the observables.

**1/** Justify

$$[\hat{S}_i(\vec{r}),\hat{S}_j(\vec{r}\,')] = \mathrm{i}\epsilon_{ijk}\,\delta(\vec{r}-\vec{r}\,')\,\hat{S}_k(\vec{r}) \qquad \text{(we set } \hbar = 1) \tag{2}$$

(with **Einstein's convention of implicit summation over repeated indices**). $\epsilon_{ijk}$ is the Levi-Civita antisymmetric tensor (cf. appendix).

Hint : For a single particle, the spin density operator is $\hat{\vec{S}}(\vec{r}) = \hat{\vec{S}}\,\delta(\vec{r}-\hat{\vec{r}})$ where $\hat{\vec{r}}$ and $\hat{\vec{S}}$ are the position and spin operators, respectively.

**2/ Equation of motion.**– Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\vec{S}}(\vec{r},t) = \int \mathrm{d}\vec{r}\,'\, v(\vec{r}-\vec{r}\,')\,\hat{\vec{S}}(\vec{r}\,',t)\times\hat{\vec{S}}(\vec{r},t) - \mathrm{i}\,v(0)\,\hat{\vec{S}}(\vec{r},t)\,, \tag{3}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\hat{S}_i(\vec{r},t) = \mathrm{i}[\hat{H}_0,\hat{S}_i(\vec{r},t)]$. Check that the result is Hermitian.

Hint : We recall $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$ (cf. appendix for further properties of $\epsilon_{ijk}$).

**3/ Magnetic susceptibility** (IMPORTANT QUESTION)**.**– The system is submitted to an external magnetic field $\vec{\mathcal{B}}(\vec{r},t)$, leading to the perturbation Hamiltonian :

$$\hat{H}_{\mathrm{pert}}(t) = -\int \mathrm{d}\vec{r}\,\hat{\vec{S}}(\vec{r})\cdot\vec{\mathcal{B}}(\vec{r},t) \tag{4}$$

Magnetic susceptibility is defined by

$$\langle S_i(\vec{r},t)\rangle_{\mathcal{B}} \stackrel{\mathrm{def}}{=} \langle S_i(\vec{r})\rangle + \int \mathrm{d}t'\mathrm{d}\vec{r}\,'\,\chi_{ij}(\vec{r}-\vec{r}\,',t-t')\,\mathcal{B}_j(\vec{r}\,',t') + O(\mathcal{B}^2) \tag{5}$$

where $\langle\cdots\rangle$ and $\langle\cdots\rangle_{\mathcal{B}}$ are quantum/statistical averages at equilibrium and out-of-equilibrium, respectively. Express the susceptibility as an equilibrium correlation function.

1. H. Bruus & K. Flensberg, *Many-body quantum theory in condensed matter physics*, Oxford University Press (2004).

**4/** Using the equation of motion, compute $\frac{\mathrm{d}}{\mathrm{d}t}\chi_{ij}(\vec{r}-\vec{r}\,',t)$.

**5/** We now assume that the interaction is such that the low temperature phase is ferromagnetic. We denote by $\vec{M}$ the average magnetization $\vec{M} \stackrel{\text{def}}{=} \langle\hat{\vec{S}}(\vec{r})\rangle$ and introduce the operator $\vec{m}(\vec{r})$ describing magnetization fluctuations ($||\vec{m}|| \ll ||\vec{M}||$) :

$$\hat{\vec{S}}(\vec{r}) = \vec{M} + \hat{\vec{m}}(\vec{r}) \tag{6}$$

Neglecting $||\vec{m}||^3$ terms in the equation derived in question **4**, deduce a differential equation for the susceptibility.

**6/ Susceptibility in Fourier space.–** We define

$$\widetilde{\chi}_{ij}(\vec{q},\omega) = \int \mathrm{d}t\mathrm{d}\vec{r}\,\chi_{ij}(\vec{r},t)\,\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r}+\mathrm{i}\omega t} \tag{7}$$

Assuming that magnetization is along Oz axis, $\vec{M} = \vec{u}_z\,M_0$, we study the two components of the tensor $\chi_{xx} = \chi_{yy}$ and $\chi_{xy} = -\chi_{yx}$ (i.e. we assume isotropy). Show that they obey the coupled differential equations :

$$\mathrm{i}[\omega - v(0)]\,\widetilde{\chi}_{xx}(\vec{q},\omega) = M_0\,[\tilde{v}(\vec{q}) - \tilde{v}(0)]\,\widetilde{\chi}_{xy}(\vec{q},\omega) \tag{8}$$

$$\mathrm{i}[\omega - v(0)]\,\widetilde{\chi}_{xy}(\vec{q},\omega) = -M_0\,[\tilde{v}(\vec{q}) - \tilde{v}(0)]\,\widetilde{\chi}_{xx}(\vec{q},\omega) + M_0 \tag{9}$$

where $\tilde{v}(\vec{q}) \stackrel{\text{def}}{=} \int \mathrm{d}\vec{r}\,v(\vec{r})\,\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{r}}$

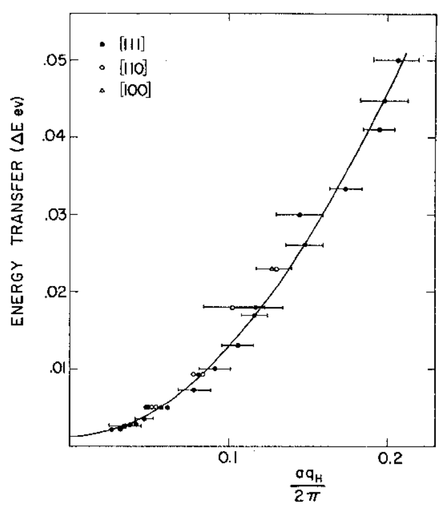


FIG. 2. The energy transfer $\Delta E$ as a function of the reduced vector of the spin waves. The error bars are about half of the full width at half maximum of the neutron groups. The solid line is the best fit to Eq. (3).

Figure 1 – *R. N. Sinclair & B. N. Brockhouse, Phys. Rev.* **120***(5), 1638 (1960). Neutron scattering on a coumpond of cobalt and 8% of iron (fcc).*

**7/ Magnons**

*a/* Solve the differential equations and show that the susceptibility diverges on a line $\omega = \omega_{\vec{q}}$. Characterize the dispersion relation $\omega_{\vec{q}}$. Recall the physical interpretation of the divergence.

*b/* Does the divergence of $\widetilde{\chi}_{ij}(\vec{q},\omega)$ really occur for $\omega \in \mathbb{R}$ ? Recall the relation between the location of the divergence and a fundamental principle.

*c/* Discuss the condition(s) for the expansion of the form : $\tilde{v}(\vec{q}) \simeq \tilde{v}(0) - \frac{A}{2}\vec{q}^{\,2}$ (Fourier transform of the potential). Relate the constant $A$ to a property of the potential. Deduce that the dispersion relation for magnons (quanta of the spin waves) is quadratic, $\omega_{\vec{q}} \underset{\vec{q}\to 0}{\simeq} \Delta_m + \frac{\vec{q}^{\,2}}{2m^*}$, where $\Delta_m$ is the excitation gap. Express the effective mass $m^*$ in terms of the potential and the magnetization.

*d/ Isotropic ferromagnet - Goldstone theorem.–* What represent magnon modes for wave vector $\vec{q} \to 0$ ? What should be the value of the parameter $v(0)$ in our model if it aims to describe an isotropic ferromagnet ?

Remark : this result is a consequence of the Goldstone theorem. In such a model, the symmetry breaking scheme is SO(3) $\to$ SO(2), i.e. the ground state has a lower symmetry, SO(2), that the Hamiltonian, SO(3), due to spontaneous symmetry breaking of the continous symmetry. This is accompanied by the emergence of *Goldstone modes*, which are collective gapless modes. [2] The physical origin of these modes is the possibility for rotating the global magnetization without energy cost.

*e/* Discuss the experimental result of Fig. 1. The experiment is done with a coumpond of Cobalt-Iron and gives $\hbar\omega_q \simeq C + \frac{1}{2}JSa^2\,\vec{q}^{\,2}$ where $JS \simeq 14.7\,\mathrm{meV}$ and $C \simeq 1.3\,\mathrm{meV}$. Assuming a lattice spacing $a \sim 1\mathring{A}$, deduce the order of magnitude for the effective mass in unit of the electon mass.

**8/ Bloch law.–** We assume that the magnetization fluctuations $\delta M(T)$ are proportionnal to the number of magnons thermally excited. Give the density of magnon modes $\rho_M(\omega)$ and deduce the behaviour of $\delta M$ with temperature $T$ and $M$.

Appendix : Levi-Civita tensor $\epsilon_{ijk}$ :
$\epsilon_{ijk}$ is the fully antisymmetric tensor (under exchange of any couple of indices) : $\epsilon_{ijk} = -\epsilon_{jik}$, etc, with $\epsilon_{123} = +1$.
Properties : $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, $\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}$ & $\epsilon_{ijk}\epsilon_{ijk} = 6$.

2. ne pas confondre la notion de mode *massif* avec la masse effective $m^*$ introduite ci-dessus. Dans la terminologie standard, un mode massif correspond à des excitations avec un gap fini. L'origine de cette terminologie vient de la structure de la fonction de Green $\langle r \,|\, \frac{1}{-\Delta+m^2} \,|\, 0 \rangle$ associée à une relation de dispersion $k^2+m^2$. La quantité $m$ joue le rôle de gap dans le spectre de l'opérateur. Dans cette théorie simple, non relativiste, $\mathrm{i}\partial_t\phi = (-\Delta + m^2)\phi$, ou relativiste, $\partial_t^2\phi = (-\Delta + m^2)\phi$, le gap, $\Delta_m^{(\text{non rel})} = m^2$ ou $\Delta_m^{(\text{rel})} = m$, est relié à la courbure à l'origine, i.e. la masse ; cependant cela n'est pas nécessairement le cas dans des modèles plus complexes.