Asymmetric Lévy flights in the presence of absorbing boundaries

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Introduction

One striking feature of Lévy flights is that their statistical behavior is dominated by a few rare and very large events, whose occurrence is thus governed by the tail of the jump distribution. This feature led us to an interesting property of the probability density function of the walker position far from the absorbing walls [1].

Power-law distributions have many applications in Physics: laser-cooling of cold atoms, random matrices, disordered systems. Recently the asymmetric Lévy flights have found applications in search problems and finance.

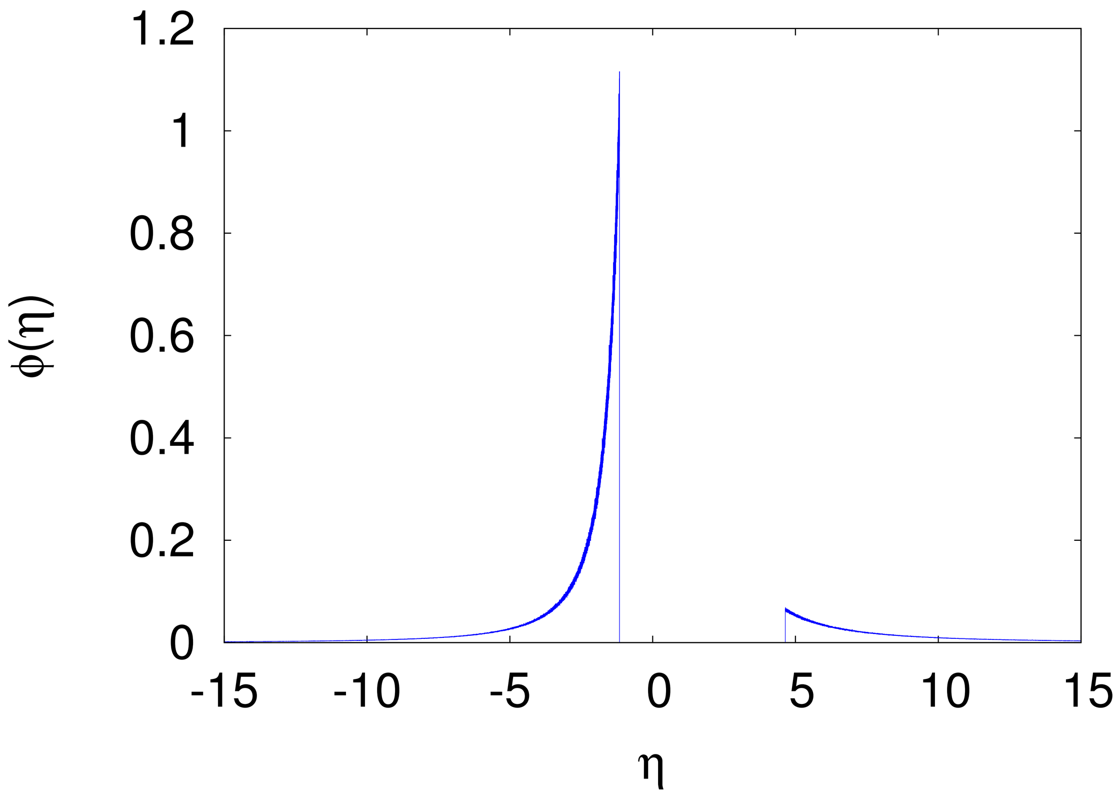
Definitions - One-dimensional Lévy walker

We consider a one-dimensional random walker, in discrete time, moving on a continuous line. Its position $x(n)$ after n steps evolves according to

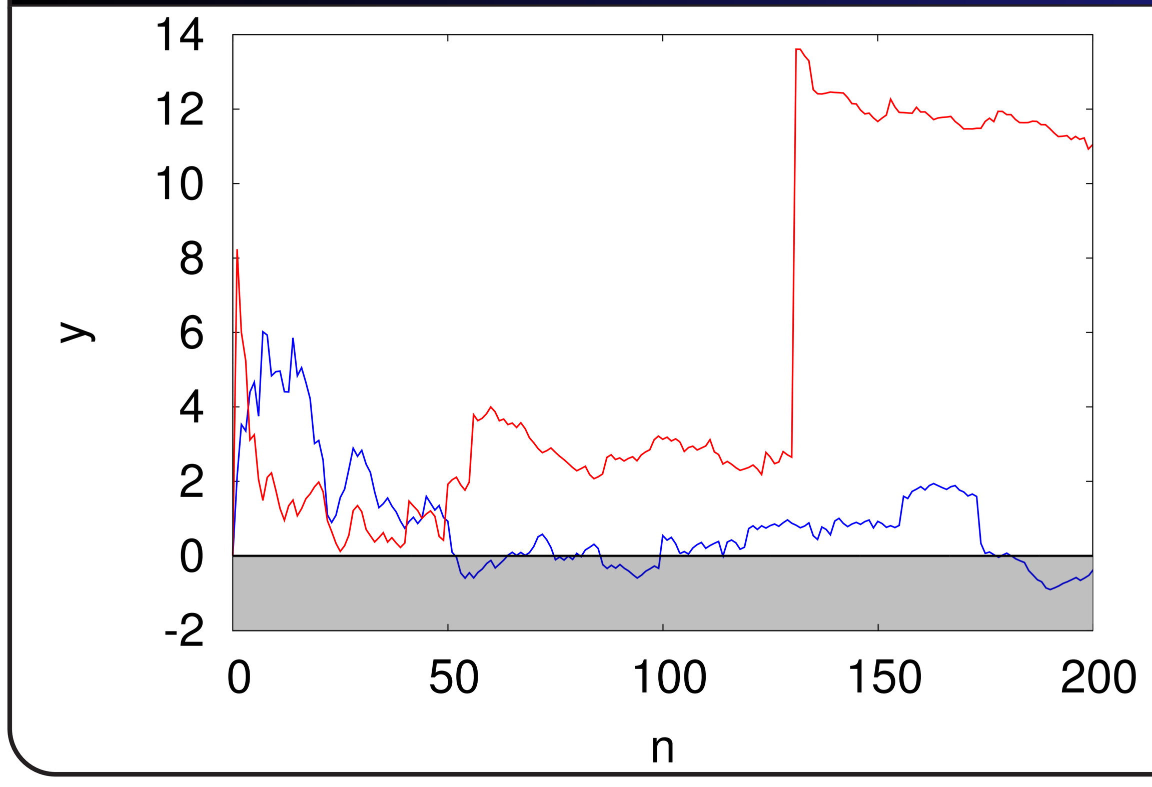
$$\begin{cases} x(0) = 0 \\ x(n) = x(n-1) + \eta(n) \end{cases} \qquad (1)$$

The random jumps variables η_i's are independent and identically distributed according to a probability density function $\phi(\eta)$ displaying asymmetric power law tails:

$$\phi(\eta) \sim \begin{cases} \dfrac{c}{\eta^{1+\alpha}} \,, \; \eta \to +\infty \,, \\ \dfrac{c/\gamma}{|\eta|^{1+\alpha}} \,, \; \eta \to -\infty \,. \end{cases} \qquad (2)$$



1D Lévy walkers



Scaling form

To study the large n behavior we write the walker position after n steps in the scaling form

$$x_n = y\, n^{1/\alpha} \,. \qquad (3)$$

When $n \to \infty$, the fluctuations of the variable y converge to a PDF which

- is independent of n and of the details of $\phi(\eta)$
- depends only on α, c and γ.

Free Lévy walker

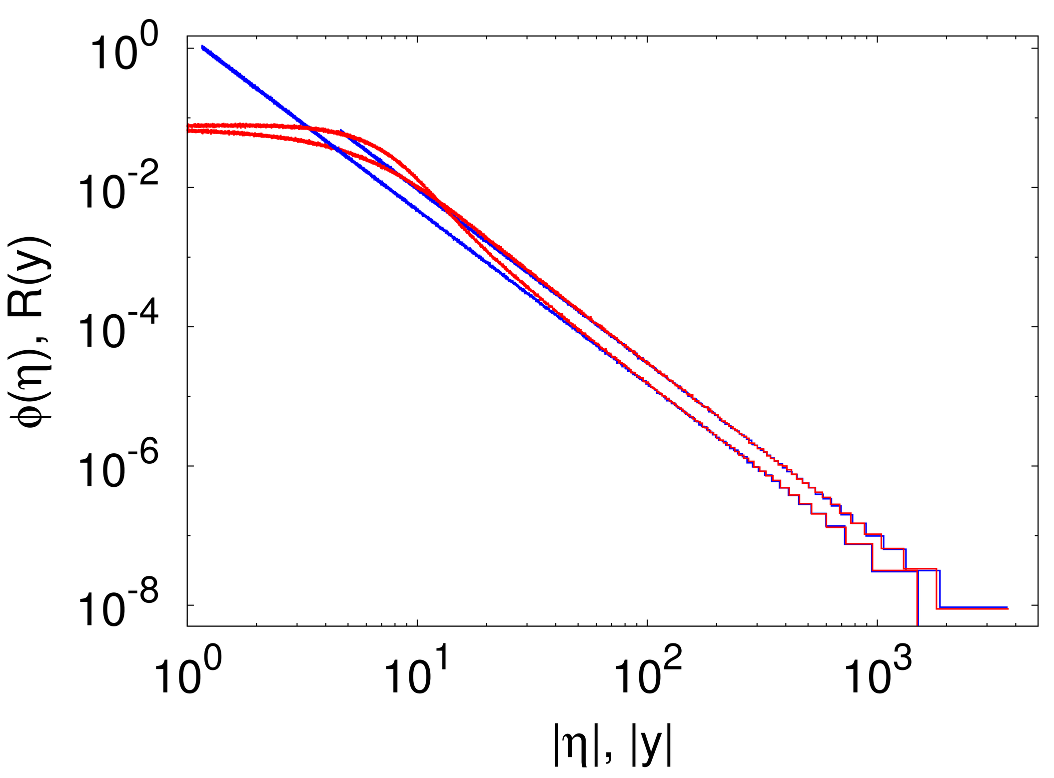
We know, from the Central Limit Theorem, that this PDF corresponds to the skewed α-stable distribution, $R(y)$, which admits the exact asymptotic expansion :

$$R(y) \sim \begin{cases} \dfrac{c}{y^{1+\alpha}} \,, \; y \to +\infty \,, \\ \dfrac{c/\gamma}{|y|^{1+\alpha}} \,, \; y \to -\infty \,, \end{cases} \qquad (4)$$

- $\alpha \in (0,2)$ is the stability index,
- $\gamma \in (0,+\infty)$ is a skewness parameter describing the asymmetry of $R(y)$,
- $c > 0$ is a scale parameter.

$R(y)$ inherits from the jump distribution $\phi(\eta)$:

- the power law tail $\propto |y|^{-\alpha-1}$,
- the amplitudes of the right and the left tails.



Constrained Lévy walker

We consider a one-dimensional random walk in presence of an absorbing wall in the negative half line, such that the walker is constrained to stay positive.

Survival Probability

The survival probability, $q_+(n)$, is the probability that the walker is still alive after n steps:

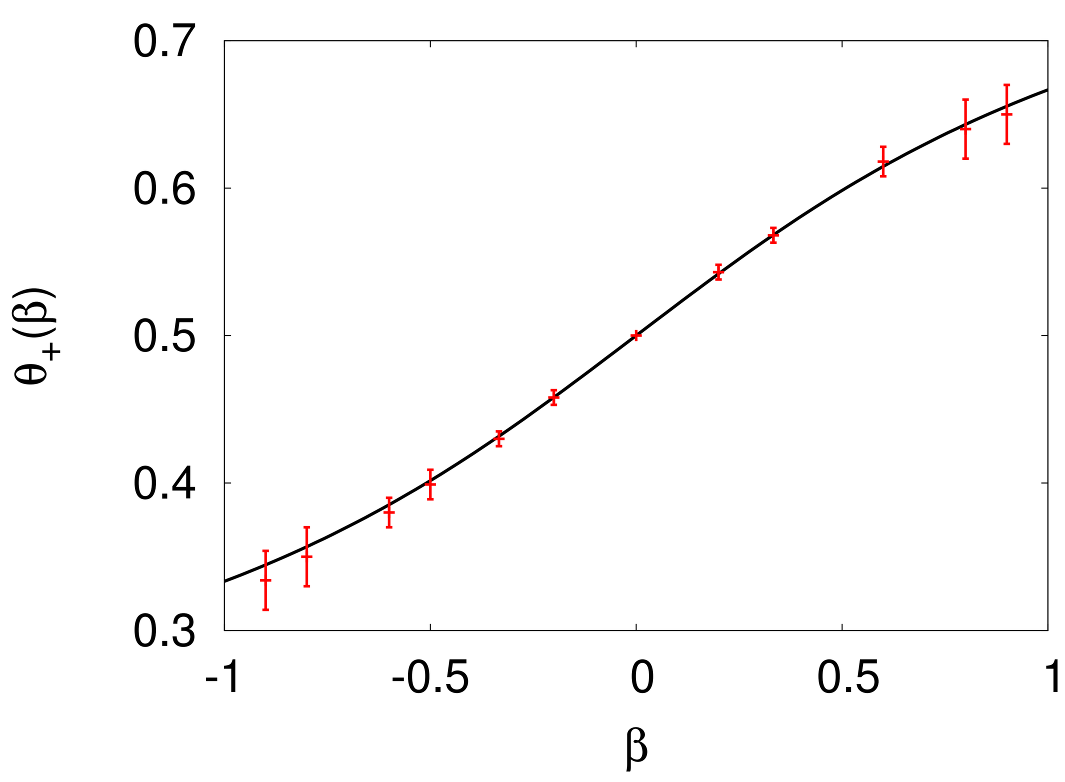
$$q_+(n) = \text{Prob.}[x(n) \geq 0, \cdots, x(1) \geq 0 | x(0) = 0] \,. \qquad (5)$$

For large n, $q_+(n)$ decays algebraically with a persistence exponents θ_+ [2]:

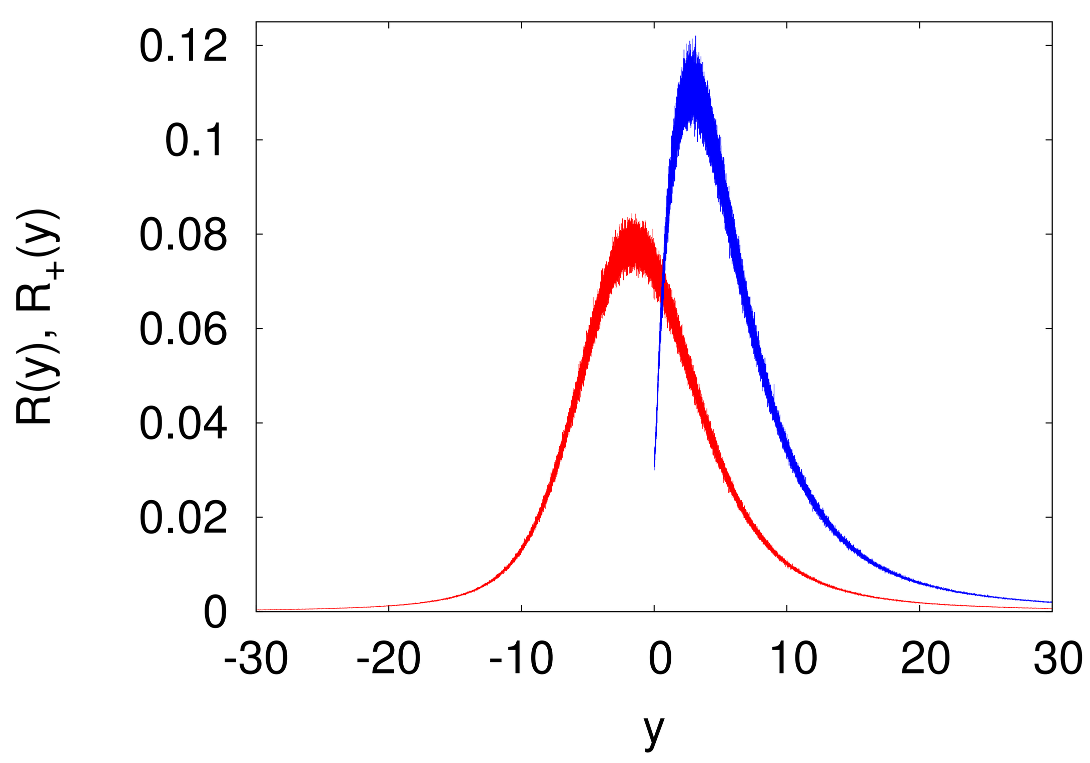
$$q_+(n) \underset{n\to\infty}{\propto} n^{-\theta_+} \,. \qquad (6)$$

Using a generalized version of the Sparre Andersen theorem we obtain the exact results for the persistence exponents (see also [3]):

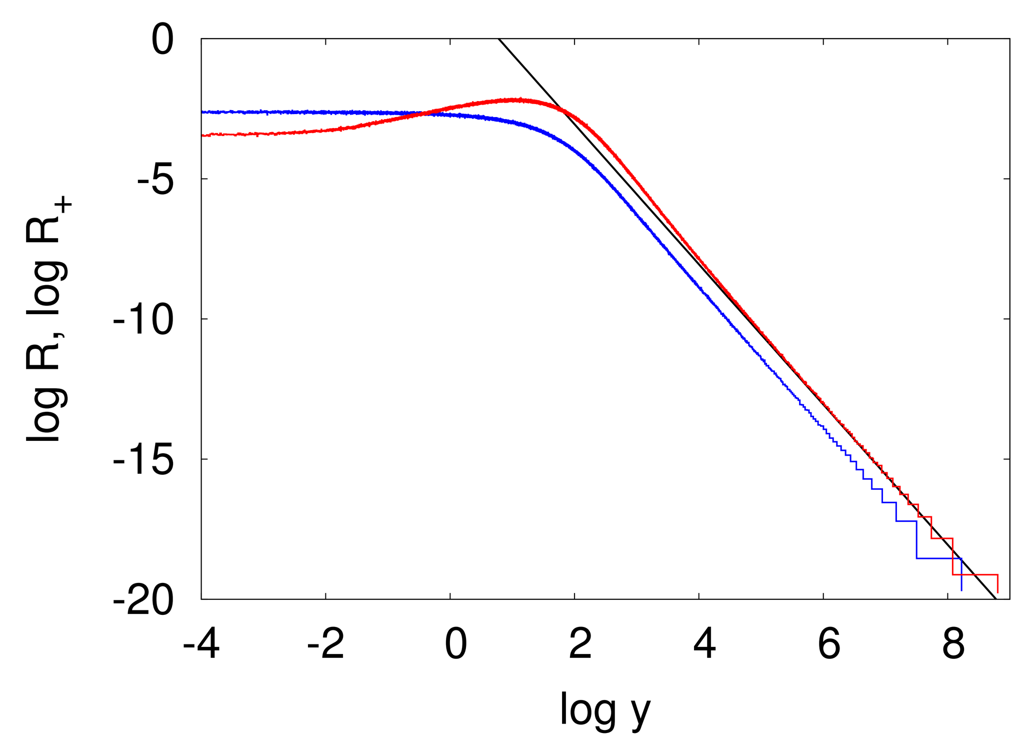
$$\theta_+ = \frac{1}{2} - \frac{1}{\pi\alpha} \arctan\left(\beta \tan\left(\frac{\pi\alpha}{2}\right)\right) \,, \quad \alpha \neq 1 \,. \qquad (7)$$



Probability Density Function



$R(y)$ and $R_+(y)$ both decay as $y^{-\alpha-1}$ when y becomes large, but with different amplitudes [4]:



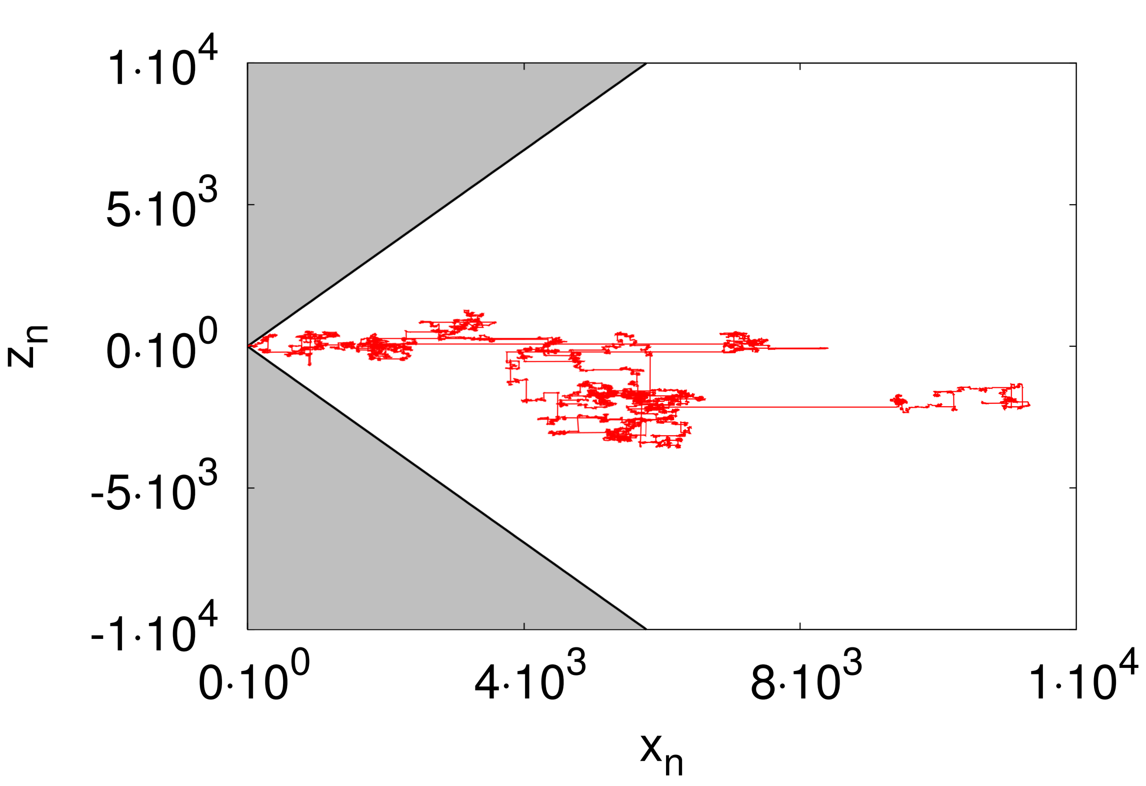
Here we compute the exact value of the amplitude c_+ and show that it is related to the corresponding persistence exponent θ_+ given in (7):

$$R_+(y) \underset{+\infty}{\sim} \frac{c_+}{y^{1+\alpha}} \,, \quad c_+ = \frac{c}{1-\theta_+} \,. \qquad (8)$$

This is in agreement with a previous result, $c_+ = 2\,c$, valid only for symmetric Lévy flights [5–7].

2D constrained Lévy walker

We now consider a two-dimensional walker constrained to stay in a semi-bounded domain $\mathcal{D}$.



In this case the survival probability has also an algebraic decay with a persistence exponent $\theta_{\mathcal{D}}$.

Far from the boundaries the PDF, $R_{d,\mathcal{D}}$, of the rescaled variable $\vec{y}$, displays the same algebraic decay as the PDF R_d, in the absence of boundaries. We then generalised the result (8):

$$\frac{R_{d,\mathcal{D}}(\vec{y})}{R_d(\vec{y})} \underset{d(\vec{y},\partial\mathcal{D})\to\infty}{\longrightarrow} \frac{1}{1-\theta_{\mathcal{D}}} \,, \qquad \text{if } \theta_{\mathcal{D}} < 1 \,. \qquad (9)$$

Every result of this poster is confirmed by careful numerical simulations.

References

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