Supplemental Material: Mixed bubbles in Bose-Bose mixtures

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THE BOGOLIUBOV VACUUM ENERGY $E_{\rm B}$ (LEE-HUANG-YANG TERM)

Case $D = 3$

In three dimensions we have

$$E_{\rm B}^{(3d)} = \frac{8}{15\pi^2}\left(\frac{m_1}{\hbar^2}\right)^{3/2}(g_{11}n_1)^{5/2} f^{(3d)}\left(\frac{m_2}{m_1}, \frac{g_{12}^2}{g_{11}g_{22}}, \frac{g_{22}n_2}{g_{11}n_1}\right), \tag{S1}$$

where

$$f^{(3d)}(z,u,x) = \frac{15}{32}\int_0^\infty \left[\frac{1}{\sqrt{2}}\sum_\pm \sqrt{k^2+\frac{xk^2}{z}+\frac{k^4}{4}+\frac{k^4}{4z^2}\pm\sqrt{\left(k^2-\frac{xk^2}{z}+\frac{k^4}{4}-\frac{k^4}{4z^2}\right)^2+\frac{4xuk^4}{z}}} - \frac{1+z}{2z}k^2 - 1 - x + \left(1+x^2z+\frac{4xzu}{1+z}\right)\frac{1}{k^2}\right]k^2dk. \tag{S2}$$

In order to calculate the integral we change the integration variable by using

$$k^2 = \frac{4\sqrt{xuz^3}}{z^2-1}\left[t-\frac{1}{t}+\frac{x-z}{\sqrt{xuz}}\right] = \frac{4\sqrt{xuz^3}}{z^2-1}\frac{(t-b_1)(t-b_2)}{t}. \tag{S3}$$

Assuming $z > 1$ and $b_2 > b_1$, the new integration interval is $t \in [b_2, \infty)$. The change of variable (S3) removes the internal square root in Eq. (S2) leading to

$$\sqrt{k^2+\frac{xk^2}{z}+\frac{k^4}{4}+\frac{k^4}{4z^2}+\sqrt{\left(k^2-\frac{xk^2}{z}+\frac{k^4}{4}-\frac{k^4}{4z^2}\right)^2+\frac{4xuk^4}{z}}} = \frac{\sqrt{8xuz}}{z^2-1}\frac{\sqrt{(t-b_1)(t-b_2)(t-a_1)(t-a_2)}}{t} \tag{S4}$$

and

$$\sqrt{k^2+\frac{xk^2}{z}+\frac{k^4}{4}+\frac{k^4}{4z^2}-\sqrt{\left(k^2-\frac{xk^2}{z}+\frac{k^4}{4}-\frac{k^4}{4z^2}\right)^2+\frac{4xuk^4}{z}}} = \frac{z\sqrt{8xuz}}{z^2-1}\frac{\sqrt{(t-b_1)(t-b_2)(t-a_1/z^2)(t-a_2/z^2)}}{t}, \tag{S5}$$

where a_1 and a_2 are roots of

$$t^2 + \frac{\sqrt{z}(xz-1)}{\sqrt{xu}}t - z^2 = 0. \tag{S6}$$

The integration of Eq. (S2) then results in a combination of elementary and elliptic functions. In the particular case $g_{12} = \pm\sqrt{g_{11}g_{22}}$ we obtain

$$\begin{aligned} f^{(3d)}(z,1,x) &= (-2-7xz+2z^2+x^2z^2)\frac{\sqrt{x+z}}{2\sqrt{z}(z^2-1)} \\ &+(-2-7xz+3z^2+3x^2z^2-7xz^3-2x^2z^4)\frac{E[\arcsin(1/z)|-xz]-E(-xz)}{2(z^2-1)^{3/2}} \\ &+(2+8xz-3z^2+6x^2z^2-2xz^3+x^2z^4)\frac{F[\arcsin(1/z)|-xz]-K(-xz)]}{2(z^2-1)^{3/2}}. \end{aligned} \tag{S7}$$

In Eq. (S7) $E(\phi|v)$ is the elliptic integral of the second kind, $E(v)$ is the complete elliptic integral, $F(\phi|v)$ is the elliptic integral of the first kind, and $K(v)$ is the complete elliptic integral of the first kind.

Finally, let us mention the identity $f^{(3d)}(z,u,x) = z^{3/2}x^{5/2}f^{(3d)}(1/z,u,1/x)$ which follows from Eq. (S1) and which can be useful, for instance, for analyzing the vicinity of the extreme limit $n_1 \to 0$, where x diverges.

Case $D = 2$

In two dimensions we have

$$E_{\rm B}^{(2d)} = \frac{1}{4\pi}\left(\frac{m_1}{\hbar^2}\right)(g_{11}n_1)^2 f^{(2d)}\left(\frac{m_2}{m_1}, \frac{g_{12}^2}{g_{11}g_{22}}, \frac{g_{22}n_2}{g_{11}n_1}, \frac{\kappa}{\sqrt{m_1 g_{11} n_1}}\right), \tag{S8}$$

where

$$f^{(2d)}(z,u,x,\tilde{\kappa}) = \int_0^{\tilde{\kappa}} \left[\frac{1}{\sqrt{2}}\sum_{\pm}\sqrt{k^2+\frac{xk^2}{z}+\frac{k^4}{4}+\frac{k^4}{4z^2}\pm\sqrt{\left(k^2-\frac{xk^2}{z}+\frac{k^4}{4}-\frac{k^4}{4z^2}\right)^2+\frac{4xuk^4}{z}}}-\frac{1+z}{2z}k^2-1-x\right]kdk. \tag{S9}$$

This function satisfies $f^{(2d)}(z,u,x,\tilde{\kappa}) = zx^2 f^{(2d)}(1/z,u,1/x,\tilde{\kappa}/\sqrt{zx})$ and, at large $\tilde{\kappa}$, can be written as

$$f^{(2d)}(z,u,x,\tilde{\kappa}) = -\frac{1+z+4uxz+x^2z+x^2z^2}{1+z}\ln\tilde{\kappa} + h(z,u,x) + O(\tilde{\kappa}^{-2}). \tag{S10}$$

We neglect the effective-range correction $O(\tilde{\kappa}^{-2})$, which is exponentially small in terms of the expansion parameter η. For our analysis it is sufficient to set $g_{12} = \pm\sqrt{g_{11}g_{22}}$ and we make use of the explicit expression

$$\begin{aligned} h(z,1,x) &= \frac{x^2z^2+x^2z+4xz+z+1}{z+1}\ln\frac{2x^{1/4}z^{3/4}}{\sqrt{z^2-1}} + \frac{x^2z^3+4x^2z^2-x^2z+z^2-4z-1}{4(z^2-1)} - \frac{(xz-1)\sqrt{z(x+z)(1+xz)}}{z^2-1} \\ &- \frac{x^2z^5-2x^2z^3+4xz^4-4xz^3+z^4+x^2z-4xz^2+4xz-2z^2+1}{(z^2-1)^2}\ln 2 - \frac{4xz^3-z^4-4xz+2z^2-1}{(z^2-1)^2}\ln\frac{z-1}{x^{1/4}z^{3/4}} \\ &+ \frac{xz(xz^2+4z-x)}{z^2-1}\ln\frac{(x/z)^{3/4}(z-1)^{3/2}\sqrt{z+1}}{\sqrt{1+xz}-\sqrt{1+x/z}} - \frac{4xz-z^2+1}{2(z^2-1)}\ln\frac{z\sqrt{1+x/z}+\sqrt{1+xz}}{z\sqrt{1+x/z}-\sqrt{1+xz}}. \end{aligned} \tag{S11}$$

and the property (useful for the case $z < 1$)

$$h(z,1,x) = x^2z\,h(1/z,1,1/x) + \frac{x^2z^2+x^2z+4xz+z+1}{z+1}\ln\sqrt{xz}. \tag{S12}$$

Case $D = 1$

In one dimension we have

$$E_{\rm B}^{(1d)} = \frac{1}{2\pi}\left(\frac{m_1}{\hbar^2}\right)^{1/2}(g_{11}n_1)^{3/2} f^{(1d)}\left(\frac{m_2}{m_1}, \frac{g_{12}^2}{g_{11}g_{22}}, \frac{g_{22}n_2}{g_{11}n_1}\right), \tag{S13}$$

where

$$f^{(1d)}(z,u,x) = \int_0^{\infty} \left[\frac{1}{\sqrt{2}}\sum_{\pm}\sqrt{k^2+\frac{xk^2}{z}+\frac{k^4}{4}+\frac{k^4}{4z^2}\pm\sqrt{\left(k^2-\frac{xk^2}{z}+\frac{k^4}{4}-\frac{k^4}{4z^2}\right)^2+\frac{4xuk^4}{z}}}-\frac{1+z}{2z}k^2-1-x\right]dk \tag{S14}$$

satisfies $f^{(1d)}(z,u,x) = z^{1/2}x^{3/2}f^{(1d)}(1/z,u,1/x)$. In the particular case $g_{12} = \pm\sqrt{g_{11}g_{22}}$ we use the explicit expression

$$\begin{aligned} f^{(1d)}(z,1,x) &= -\frac{4}{3}\sqrt{1+\frac{x}{z}} + \frac{4}{3}\frac{1}{\sqrt{z^2-1}}\Big\{(xz-1)E[\arcsin(1/z)|-xz] + (xz+1)F[\arcsin(1/z)|-xz] \\ & -2(xz-1)[E(-xz)-(xz+1)K(-xz)] + \sqrt{xz+1}\left[(xz-1)E\left(\frac{xz}{xz+1}\right) - (2xz-1)K\left(\frac{xz}{xz+1}\right)\right]\Big\}. \end{aligned} \tag{S15}$$

MIXED-BUBBLE REGIME BOUNDARIES

Case $D = 3$

In the upper row of Fig. 2 of the main text, the blue and orange curves comprising the blue areas are determined, respectively, from the conditions $\Omega''(n_\mathrm{L}) = 0$ and $\Omega'(n_\mathrm{R}) = [\Omega(n_\mathrm{R}) - \Omega(n_\mathrm{L})]/(n_\mathrm{R} - n_\mathrm{L})$, the same conditions as in the mass-balanced case, explained in the main text. They give, respectively,

$$\frac{\hbar^3 \delta g_{\min}(z,\alpha)}{g\sqrt{m^3 g^3 n_+}} = \frac{(\alpha^2+1)^{1/4}}{\pi^2 \alpha^{3/2} z^{3/4}} \left[\frac{-4 + 4z^2 + 12z\alpha + 3z^2\alpha^2}{4(z^2-1)} + \frac{3z^2\alpha(-4z - 2\alpha + z^2\alpha)}{4(z^2-1)^{3/2}} \arccos(1/z) \right] \tag{S16}$$

and

$$\frac{\hbar^3 \delta g_{\max}(z,\alpha)}{g\sqrt{m^3 g^3 n_+}} = \frac{(\alpha^2+1)^{1/4}}{\pi^2 \alpha^{3/2} z^{3/4}} \left[\frac{-8 + 8z^2 - 30z^{5/2}\alpha^{3/2} - 12z^{3/2}\alpha^{5/2} + 12z^{7/2}\alpha^{5/2}}{15(z^2-1)} - \frac{2(z\alpha)^{3/2}\ln(z - \sqrt{z^2-1})}{(z^2-1)^{3/2}} \right]. \tag{S17}$$

The boundaries comprising the pink areas are given, respectively, by Eqs. (S16) and (S17) with replaced $\alpha \to 1/\alpha$ and $z \to 1/z$ (equivalent to the interchange $1 \rightleftarrows 2$).

The red upper borders of the "pinch" areas are obtained by the condition that the convex lobe of $\Omega(n_-)$ has a minimum degenerate with $\Omega(n_\mathrm{L})$ and $\Omega(n_\mathrm{R})$.

Case $D = 2$

The two-dimensional case is analyzed in the same manner as the three-dimensional one. In spite of the seemingly explicit dependence on κ the results are actually cut-off independent (to the chosen approximation order) because of the running

$$g_{\sigma\sigma'} = \frac{\pi\hbar^2}{\mu_{\sigma\sigma'} \ln[2e^\gamma/\kappa a^{(2d)}_{\sigma\sigma'}]}, \tag{S18}$$

where $\mu_{\sigma\sigma'} = m_\sigma m_{\sigma'}/(m_\sigma + m_{\sigma'})$. In our notations ($z = m_2/m_1$ and $m = \sqrt{m_1 m_2}$) $\mu_{11} = m/2\sqrt{z}$, $\mu_{22} = m\sqrt{z}/2$, and $\mu_{12} = m/(\sqrt{z} + 1/\sqrt{z})$. Equation (S18) is valid for $\eta = \mu_{\sigma\sigma'} g_{\sigma\sigma'}/\hbar^2 \ll 1$, which gives the freedom to choose the cut-off scale. Indeed, choosing another cutoff $\tilde\kappa$ the new coupling constant $\tilde g_{\sigma\sigma'}$ is related to the old one by the equation

$$g_{\sigma\sigma'} \approx \tilde g_{\sigma\sigma'} - \frac{\mu_{\sigma\sigma'} \tilde g^2_{\sigma\sigma'}}{\pi\hbar^2} \ln\frac{\tilde\kappa}{\kappa} \approx \tilde g_{\sigma\sigma'} - \frac{\mu_{\sigma\sigma'} g^2_{\sigma\sigma'}}{\pi\hbar^2} \ln\frac{\tilde\kappa}{\kappa}. \tag{S19}$$

The tilde is removed in the second-order term since the difference between g^2 and $\tilde g^2$ is third order in η and can be neglected in the Bogoliubov approximation. In the same spirit the κ-dependence of $g_{\sigma\sigma'}$ in the MF energy term gets cancelled by the explicit logarithmic dependence of E_B.

From Eq. (S19) one can also derive the running of $\delta g = g_{12} - \sqrt{g_{11} g_{22}}$. Namely,

$$\delta g \approx \delta\tilde g + \frac{m\tilde g^2}{4\pi\hbar^2}\left(\frac{1}{\alpha\sqrt{z}} + \alpha\sqrt{z} - \frac{4}{\sqrt{z} + 1/\sqrt{z}}\right)\ln\frac{\tilde\kappa}{\kappa} \approx \delta\tilde g + \frac{m g^2}{4\pi\hbar^2}\left(\frac{1}{\alpha\sqrt{z}} + \alpha\sqrt{z} - \frac{4}{\sqrt{z} + 1/\sqrt{z}}\right)\ln\frac{\tilde\kappa}{\kappa}. \tag{S20}$$

The middle row in Fig. 2 thus presents the boundaries of the mixed-bubble regime in terms of the renormalized ratio

$$R = \frac{\hbar^2 \delta g}{m g^2} - \frac{1}{8\pi}\left(\frac{1}{\alpha\sqrt{z}} + \alpha\sqrt{z} - \frac{4}{\sqrt{z} + 1/\sqrt{z}}\right)\ln\frac{m g n_+}{\hbar^2\kappa^2}, \tag{S21}$$

which is κ independent in the Bogoliubov approximation. The blue and orange curves comprising the blue-shaded regions in the middle row of Fig. 2 are obtained in the same manner as Eqs. (S16) and (S17) and are given, respectively, by the formulas

$$\begin{aligned} R_{\min}(z,\alpha) &= \frac{1}{16\pi(z^2-1)\sqrt{z\alpha}}[(2 + 4\ln 2)z^3\alpha^2 + 4z^2\alpha^2 - (8 + 16\ln 2)z^2\alpha - (6 + 4\ln 2)z\alpha^2 + 8z\alpha + 4z^2 - 4 \\ &+ (3z^3\alpha^2 - 12z^2\alpha - 3z\alpha^2 - z^2 - 4z\alpha + 1)\ln z - (2z^3\alpha^2 - 8z^2\alpha - 2z\alpha^2 + 2z^2 + 8z\alpha - 2)\ln\alpha \\ &+ (z^3\alpha^2 - 4z^2\alpha - z\alpha^2 + z^2 + 4z\alpha - 1)\ln(\alpha^2 + 1) - (4z^3\alpha^2 - 16z^2\alpha - 4z\alpha^2)\ln(z + 1). \end{aligned} \tag{S22}$$

and

$$R_{\max}(z,\alpha) = \frac{3z\alpha^2 + 1 - \ln[z\alpha^2/(\alpha^2+1)]}{16\pi\sqrt{z}\alpha} + \frac{\sqrt{z}(z\alpha + \alpha - 4)\ln[z(\alpha^2+1)]}{16\pi(z+1)} - \frac{\sqrt{z}\ln[(z+1)/2]}{\pi(z^2-1)}. \tag{S23}$$

The pink-shaded areas are restricted by the curves $R_{\min}(1/z, 1/\alpha)$ (blue) and $R_{\max}(1/z, 1/\alpha)$ (orange). The red boundary is calculated numerically from the condition that $\Omega(n_-)$ has three degenerate minima.

Case $D = 1$

The analysis is also the same. The blue and orange curves for the blue-shaded regions in the lower row of Fig. 2 are obtained from the formulas,

$$\frac{\hbar\delta g_{\min}(z,\alpha)}{g\sqrt{mg/n_+}} = -\frac{1}{8\pi z^{1/4}\sqrt{\alpha}(\alpha^2+1)^{1/4}}\left[2 + \alpha^2 + \frac{z\alpha(z\alpha-4)}{\sqrt{z^2-1}}\arccos(1/z)\right] \tag{S24}$$

and

$$\frac{\hbar\delta g_{\max}(z,\alpha)}{g\sqrt{mg/n_+}} = -\frac{1}{3\pi[z\alpha^2(\alpha^2+1)]^{1/4}}\left[2 + \sqrt{z\alpha^3} + \frac{3\sqrt{z\alpha}\ln(z-\sqrt{z^2-1})}{\sqrt{z^2-1}}\right], \tag{S25}$$

and, as in higher dimensions, the pink-shaded region boundaries are obtained from Eqs. (S24) and (S25) by setting $\alpha \to 1/\alpha$ and $z \to 1/z$.