Thermal transport and quasi-particle hydrodynamics

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EDUCATION SCIENCE INNOVATION

PSL

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Phonon gas? Fermi liquid?

Quasi-particles in solids	**Hydrodynamics**
• A lattice (and its defects) • Collisions limit the flow by giving away momentum to the host solid. • Dissipation arises even in absence of viscosity.	• No lattice • Collisions conserve momentum and energy. T • Locally well-defined thermodynamics • Viscosity is the source of dissipation.
Boltzmann equation	**Navier-Stokes equation**

SOVIET PHYSICS USPEKHI VOLUME 11, NUMBER 2 SEPTEMBER–OCTOBER, 1968

530.145 + 536.48

HYDRODYNAMIC EFFECTS IN SOLIDS AT LOW TEMPERATURE

R. N. GURZHI

Physico-technical Institute, Academy of Sciences, Ukrainian SSR, Khar'kov

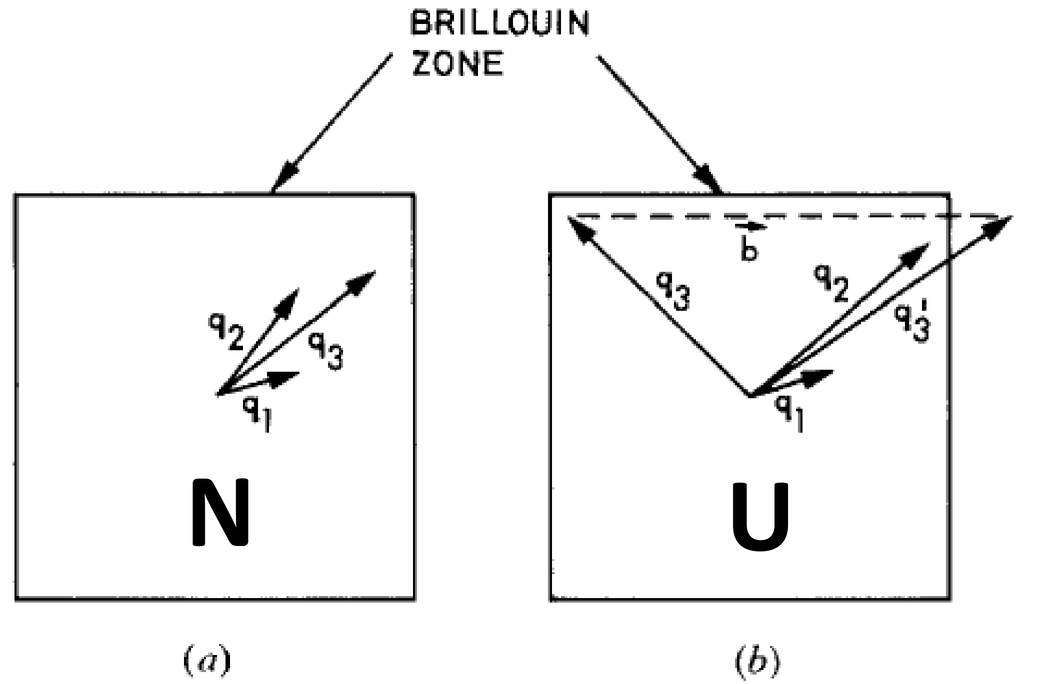
Usp. Fiz. Nauk 94, 689–718 (April, 1968)

"The phenomena of thermal conductivity of insulators and the electrical conductivity of metals have specific properties.
In both cases the total quasi-particle current turns out to be non-vanishing.
It follows that when only normal collisions occur in the system, there could exist an undamped current in the absence of an external field which could sustain it."

Without umklapp collisions, finite viscosity would set the flow rate!

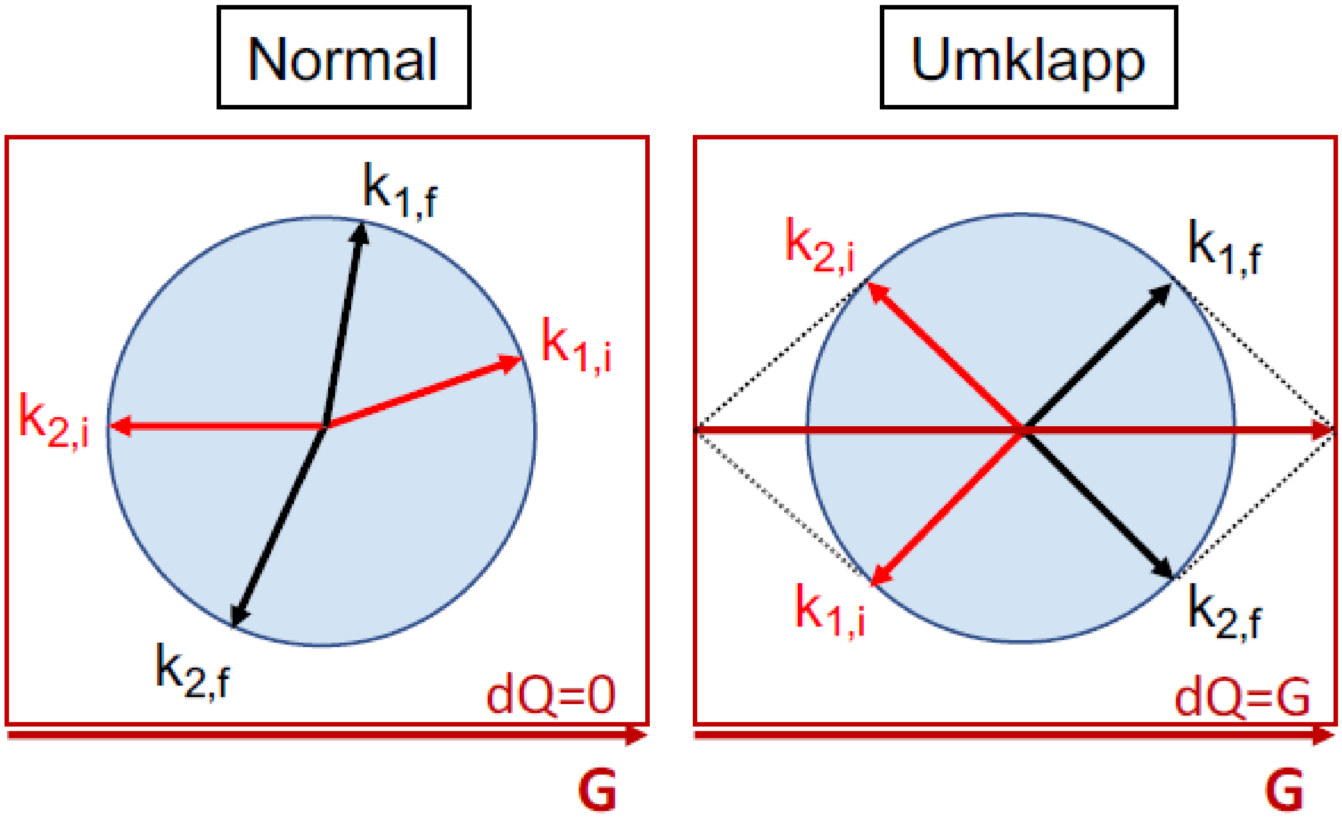
Normal and Umklapp scattering(phonons)

- **q1, q2: wave-vectors of colliding phonons**
- **q3: resultant wave vector**



U scattering events become rare at low temperature!

Normal and Umklapp scattering (electrons)



The frequency of U-scattering events depend on the size of the Fermi surface!

The Boltzmann picture

Decreasing temperature

Phonon conductivity

- Ph-Ph scattering

  $\kappa \propto T^{-1}$

- Scattering by defects

Peak in κ

- Scattering by boundaries

  $\kappa \propto T^{3}$

Electron conductivity

- Scattering by phonons at high T

  $\rho \propto T$

- Scattering by small-q phonons $\rho \propto T^{5}$

- e-e scattering

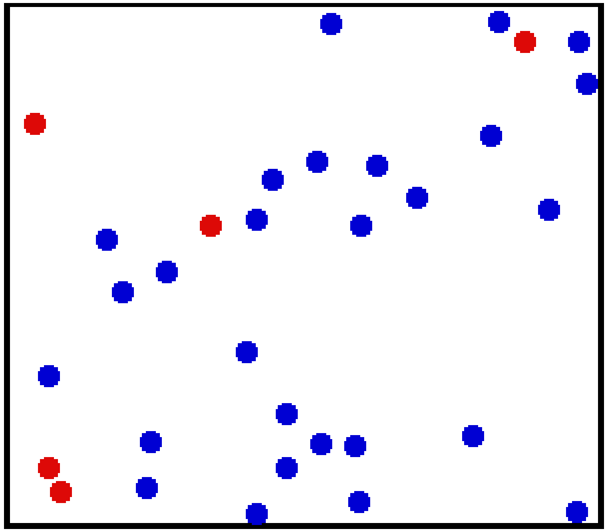
  $\rho \propto T^{2}$

- Scattering by defects and boundaries ρ_0

What about Normal collisions?

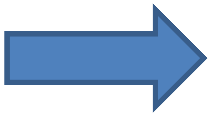
Phonons

Kinetic theory of gases



$$\kappa = 1/3\ C\ v\ l$$

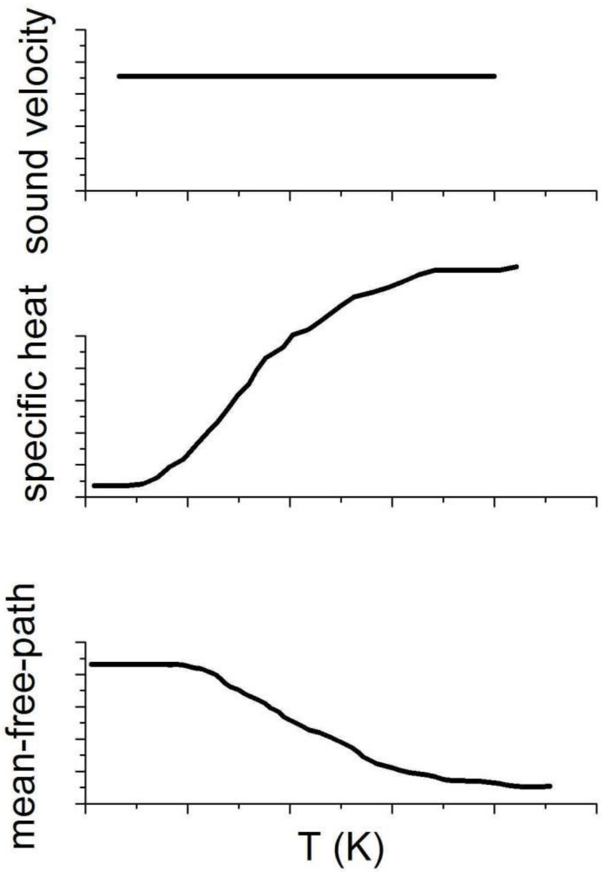
- Specific heat per volume
- Average velocity
- Mean-free-path of atomic particles

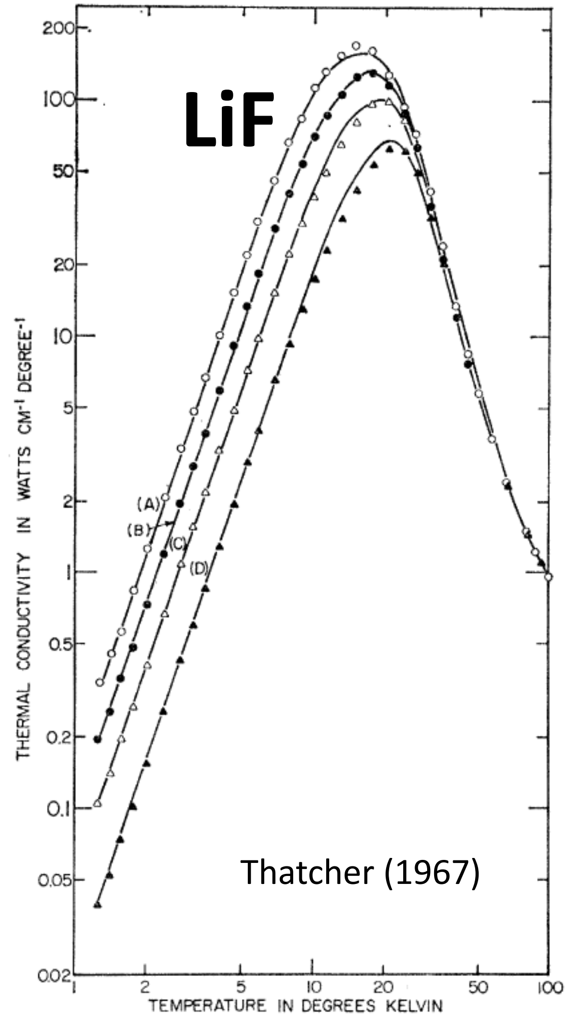


Thermal conductivity

Heat conduction in insulators

$\kappa = 1/3\ C\ v\ l$



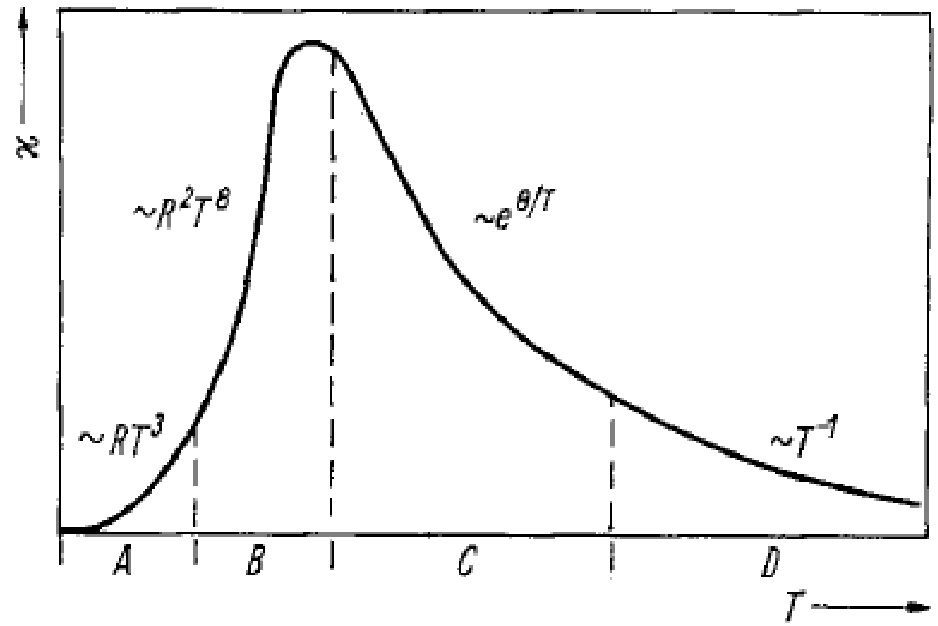


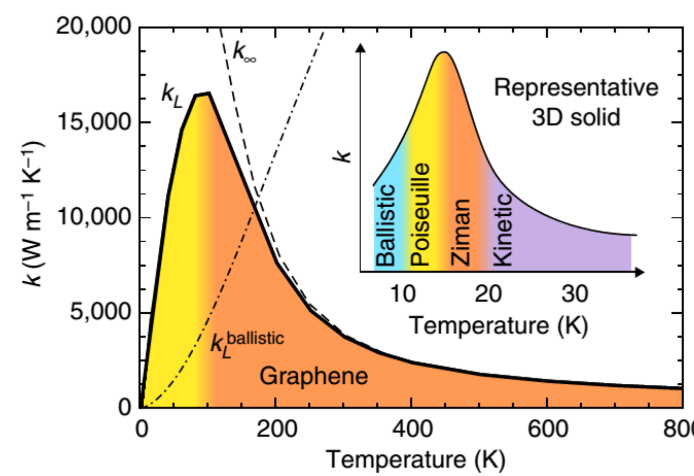
The larger the sample the higher the low-temperature thermal conductivity!

H. Beck (a)[1]), P. F. Meier (b)[2]), and A. Thellung (c)

phys. stat. sol. (a) 24, 11 (1974)

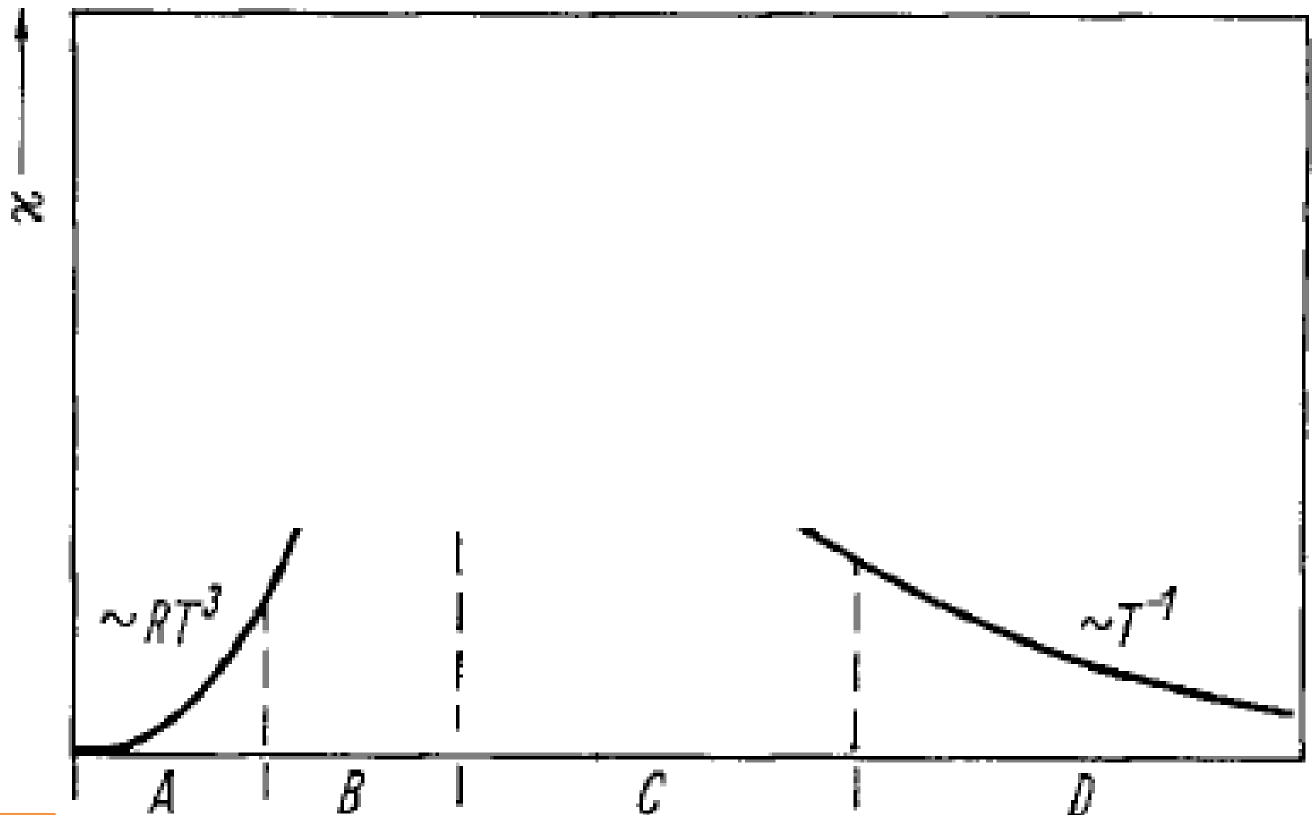
Fig. 1. Different regions of thermal conductivity:
A:
Casimir region; $\tau_B \ll \tau_N$, $\tau_B \ll \tau_R$
B:
Poiseuille flow region; $\tau_N \ll \tau_B \ll \tau_R$
C:
Ziman region; $\tau_N \ll \tau_R \ll \tau_B$
D:
kinetic region; $\tau_R \ll \tau_N \ll \tau_B$
Here τ_N, τ_R, τ_B denote the relaxation times for normal processes, resistive processes, and boundary scattering, respectively





A. Cepellotti et al., Nature Comm. 2014

Regimes of heat transport



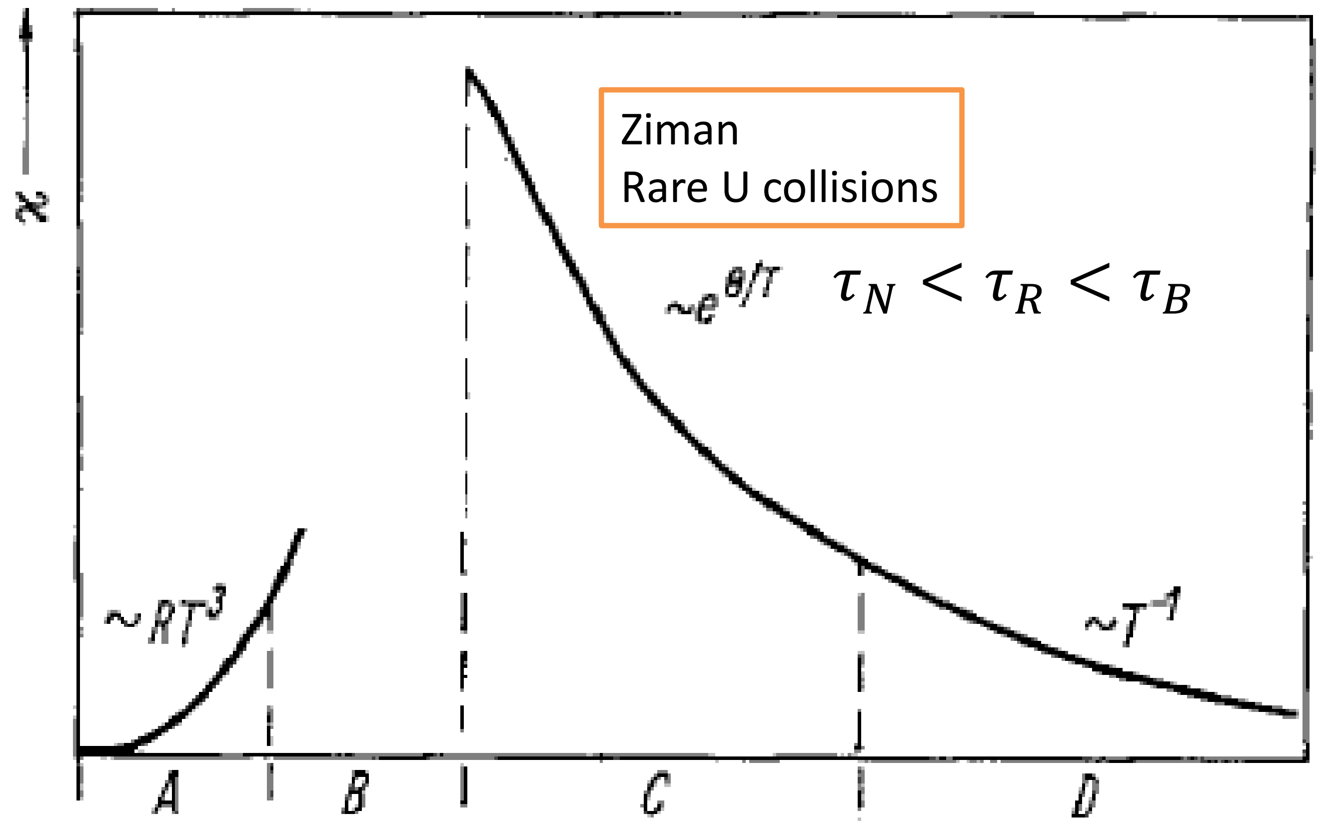
Ballistic
mfp constant

Kinetic
Abundant U collisions

$\tau_B < \tau_R, \tau_N$

$\tau_R < \tau_N < \tau_B$

Regimes of heat transport



Ballistic
mfp constant

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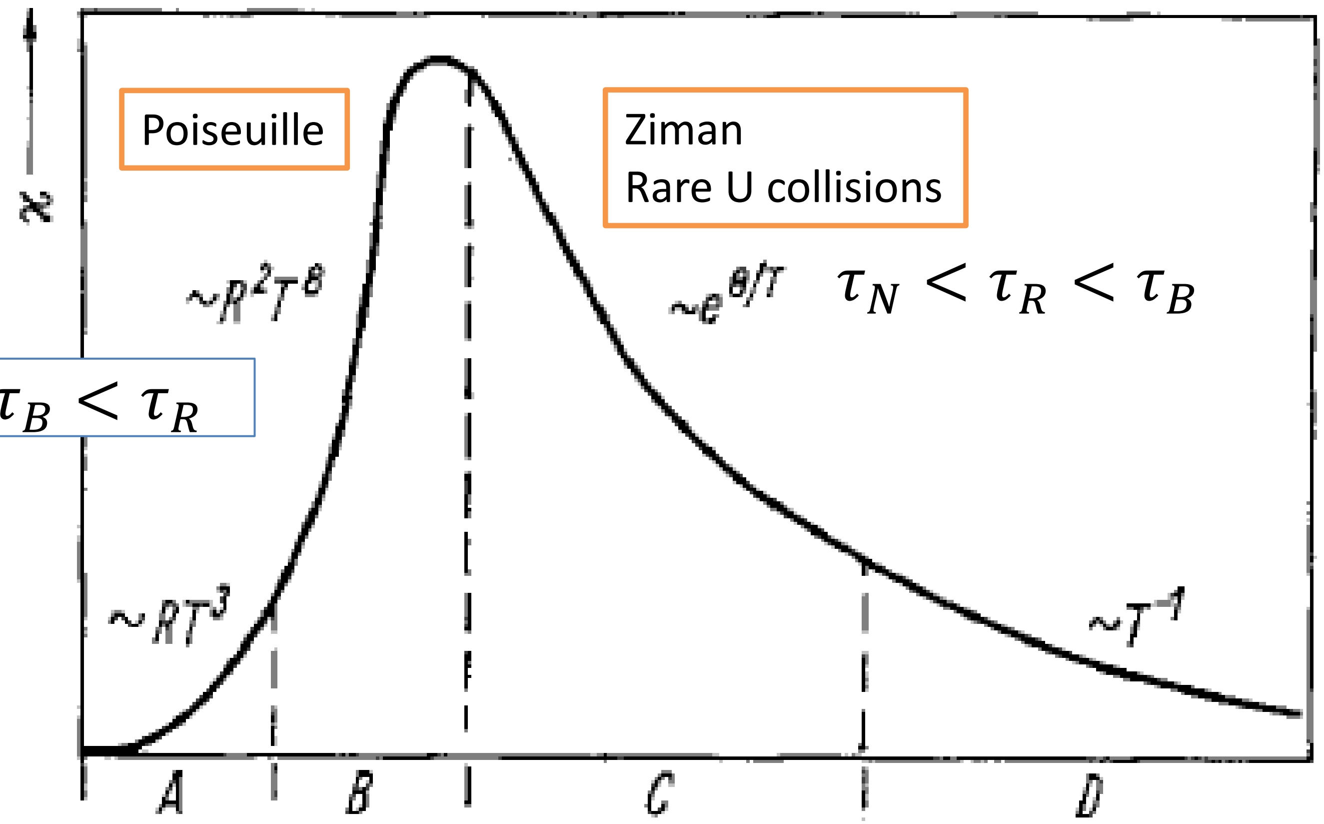
Regimes of heat transport



Poiseuille

Ziman
Rare U collisions

$\tau_N < \tau_R < \tau_B$

$\tau_N < \tau_B < \tau_R$

Ballistic
mfp constant

$\tau_B < \tau_R, \tau_N$

Kinetic
Abundant U collisions

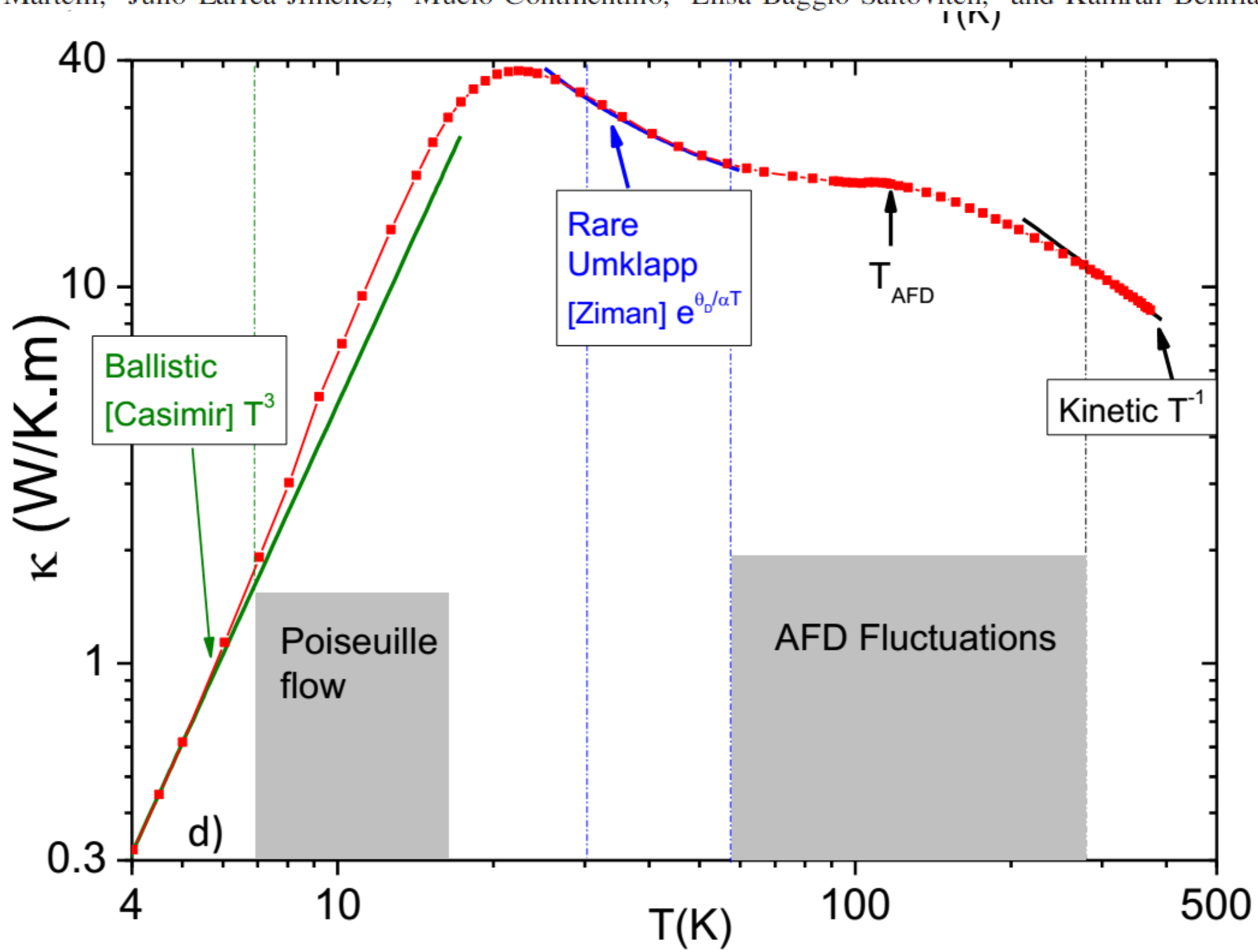
$\tau_R < \tau_N < \tau_B$

PHYSICAL REVIEW LETTERS 120, 125901 (2018)

Editors' Suggestion

Thermal Transport and Phonon Hydrodynamics in Strontium Titanate

Valentina Martelli,[1] Julio Larrea Jiménez,[2] Mucio Continentino,[1] Elisa Baggio-Saitovitch,[1] and Kamran Behnia[3,4]



Theoretical Poiseuille flow of phonons

- Predicted by Gurzhi (1959-1965)
- Expected to follow T^8!

$$\kappa = 1/3\ C\ v\ l_{eff}$$

$$\ell_{eff} = \frac{d^2}{\ell_N}$$ ← Distance between two normal collisions!

$$\ell_N \propto T^{-5}$$

$$\ell_{eff} \propto T^5$$

$$C \propto T^3$$

⇒

$$\kappa \propto T^8$$

Experimental phonon Poiseuille flow

- Diagnosed in a handful of solids!
- Whenever thermal conductivity evolves faster than specific heat!

$$\kappa \propto T^{\gamma}$$

$$\gamma > \gamma'$$

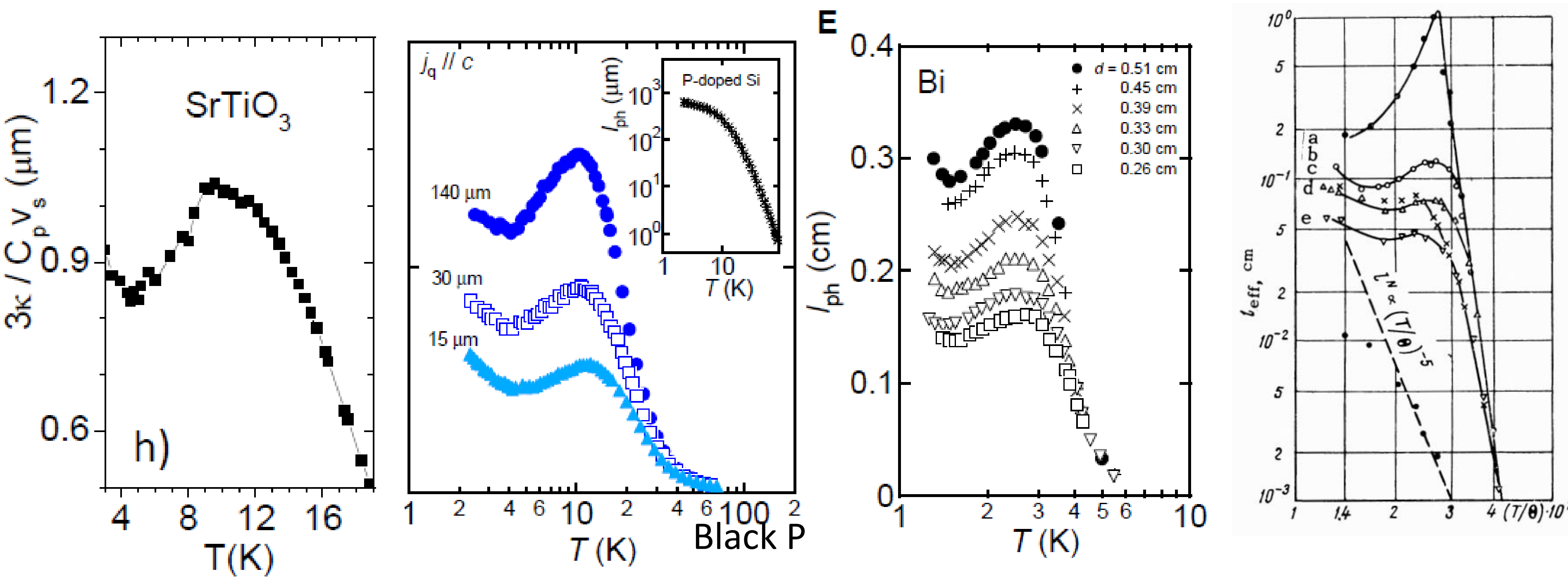
$$C \propto T^{\gamma'}$$

γ and γ' both close to 3!

- He^4 solid (Mezhov-Deglin 1965)
- He^3 solid (Thomlinson 1969)
- Bi (Kopylov 1971)
- H (Zholonko 2006)
- Black P (Machida 2018)
- $SrTiO_3$ (Martelli 2018)
- Graphite (2020 Machida)
- Sb (2021 Jaoui)

A Knudsen minimum and a Poiseuille peak

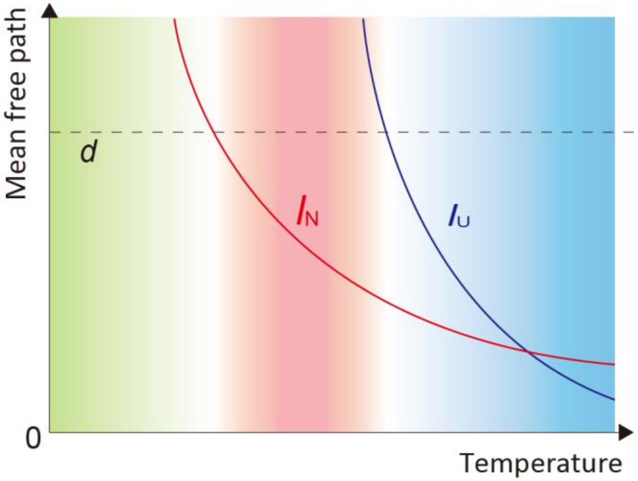
Solid He

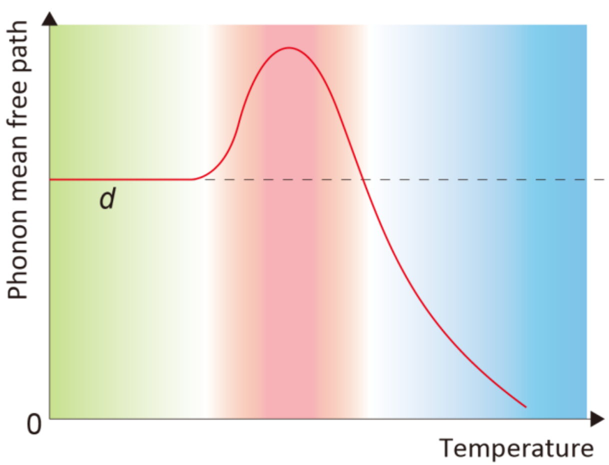


The higher the rate of momentum-conserving collisions the lower the viscosity!

Hydrodynamic

Ballistic diffusive

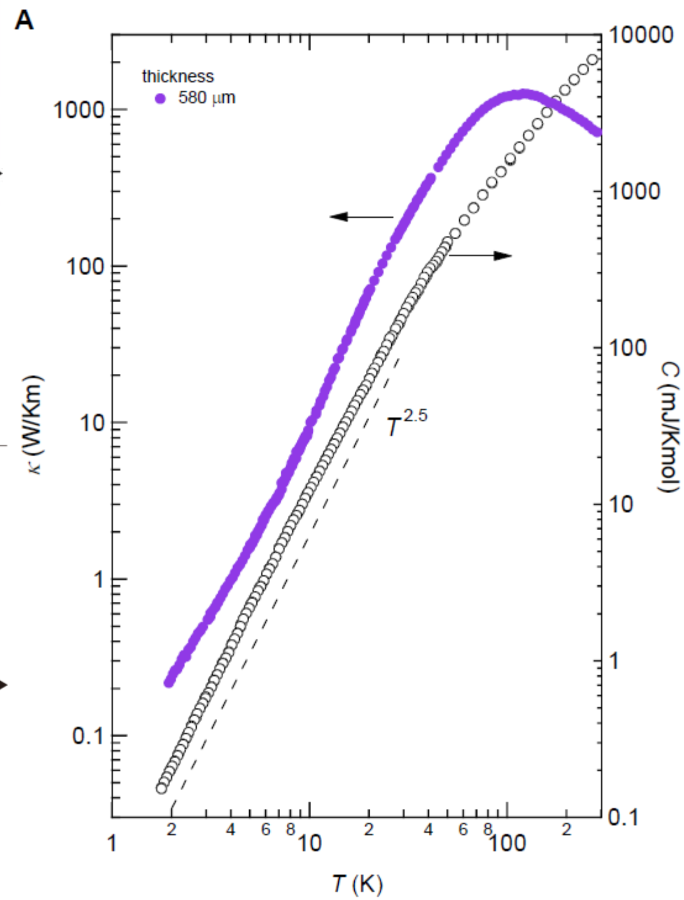


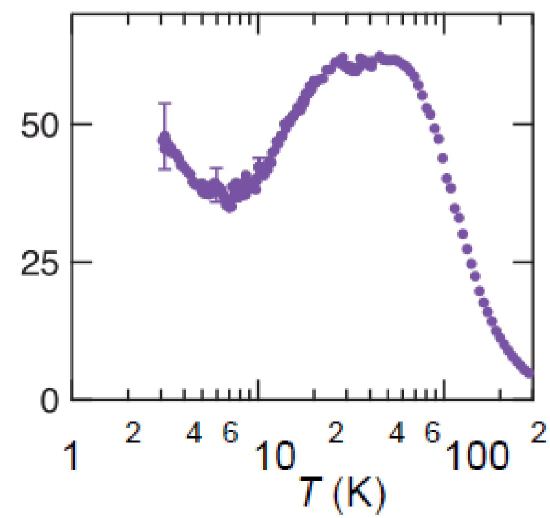


THERMAL CONDUCTIVITY

Phonon hydrodynamics and ultrahigh–room-temperature thermal conductivity in thin graphite

Yo Machida[1]*, Nayuta Matsumoto[1], Takayuki Isono[1], Kamran Behnia[2]*





Electrons

T-square resistivity

The electric resistivity of Fermi liquids follows:

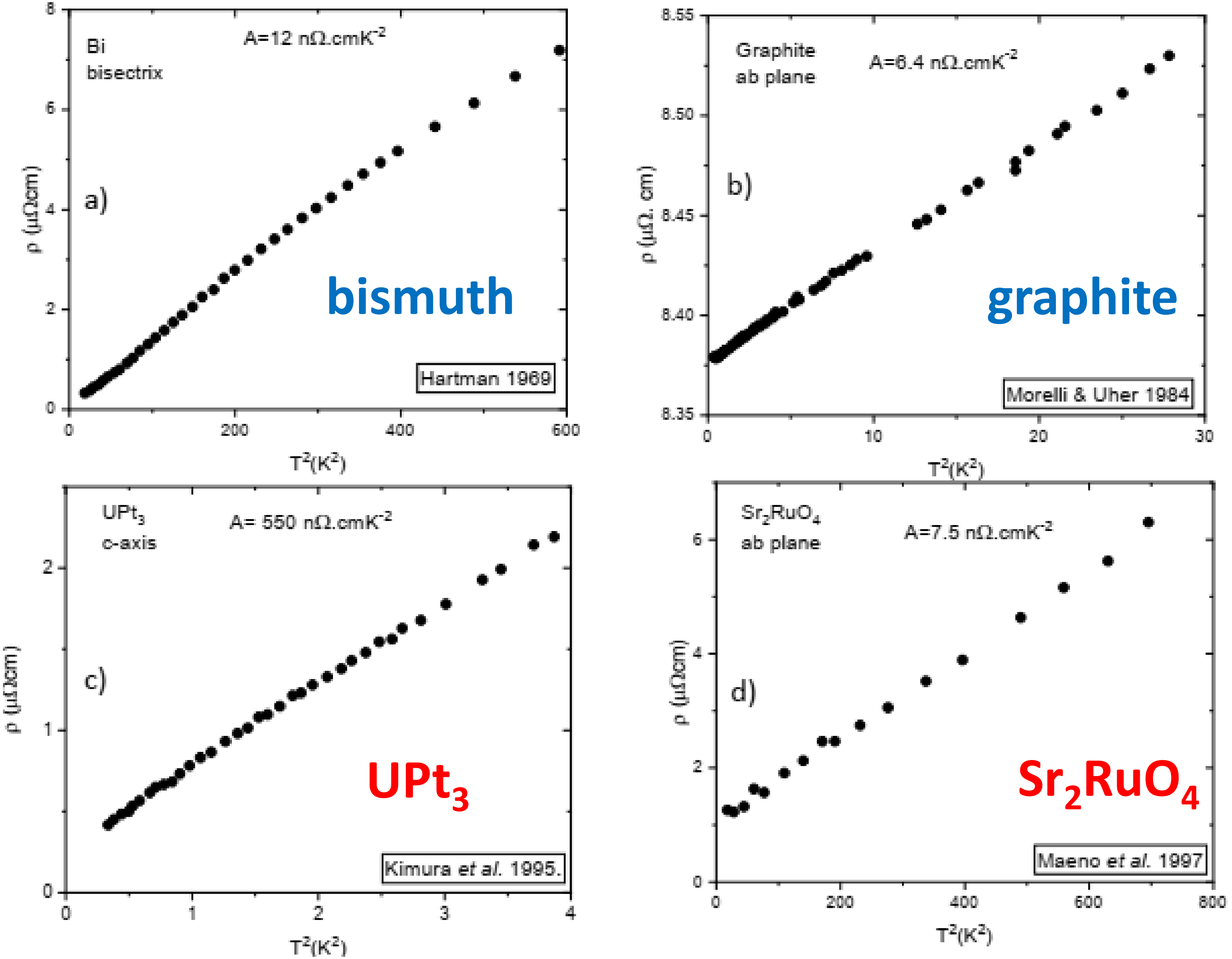
$$\rho = \rho_0 + AT^2$$

Scattering by impurities (→ ρ_0)

Electron-electron scattering (→ AT^2)

- Apply Pauli exclusion principle to each colliding electron. Then the phase space grows $\propto \left(\frac{k_B T}{E_F}\right)^2$
- Hard to see in common metals (overwhelmed by phonon scattering), but not in correlated or dilute metals.

T-square resistivity



The Wiedemann-Franz law

$$\mathrm{L} = L_0$$

Lorenz number

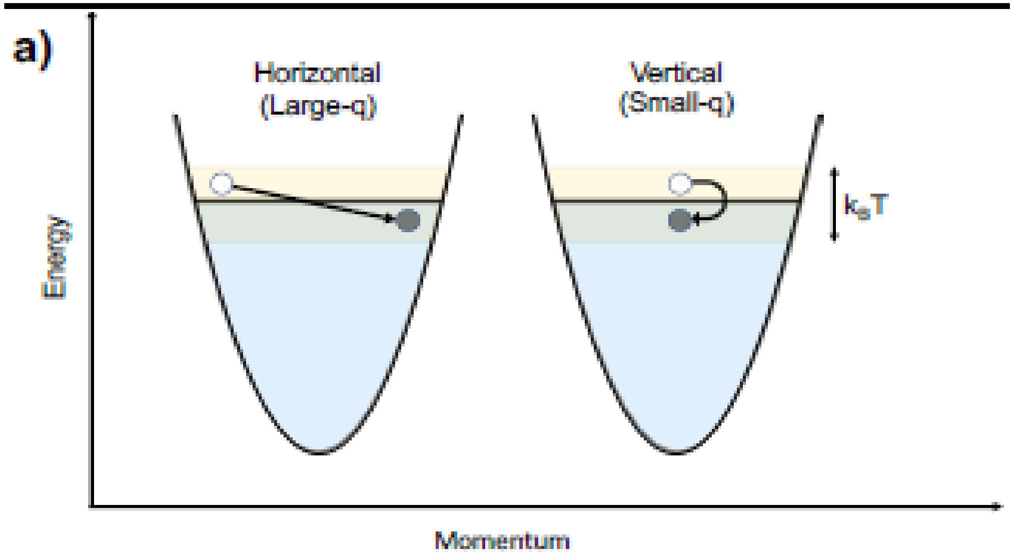
$$\mathrm{L} = \frac{\kappa}{\sigma T}$$

Sommerfeld value

$$L_0 = \frac{\pi^2}{3}\left(\frac{\kappa_B}{e}\right)^2$$

$= 2.445\ 10^{-8}\ \mathrm{W}\ \Omega\ /\ \mathrm{K}^2$

Valid at T=0



Finite-temperature deviation due to vertical scattering!

REPORTS

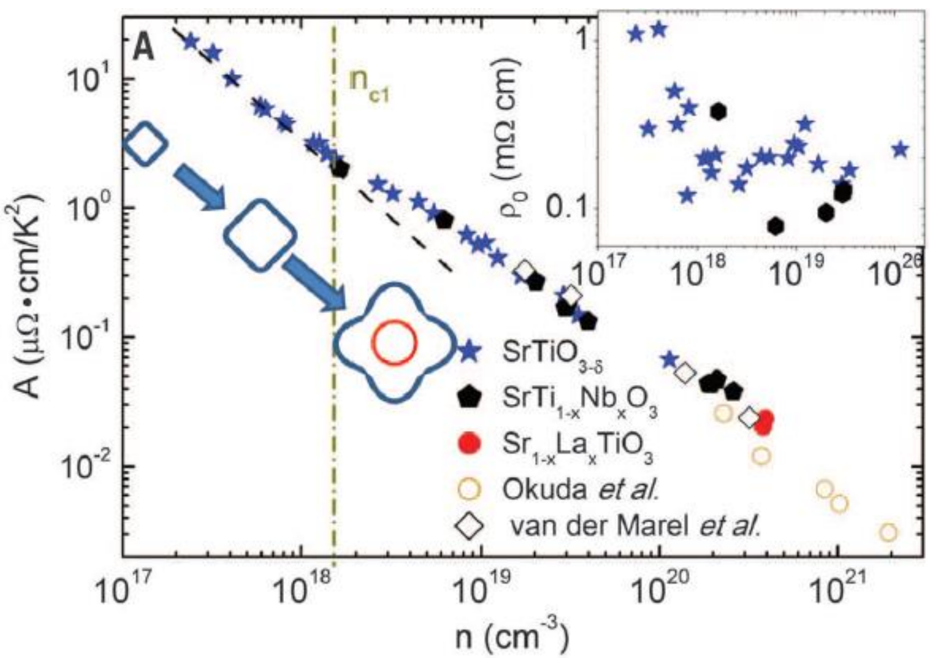
SOLID-STATE PHYSICS

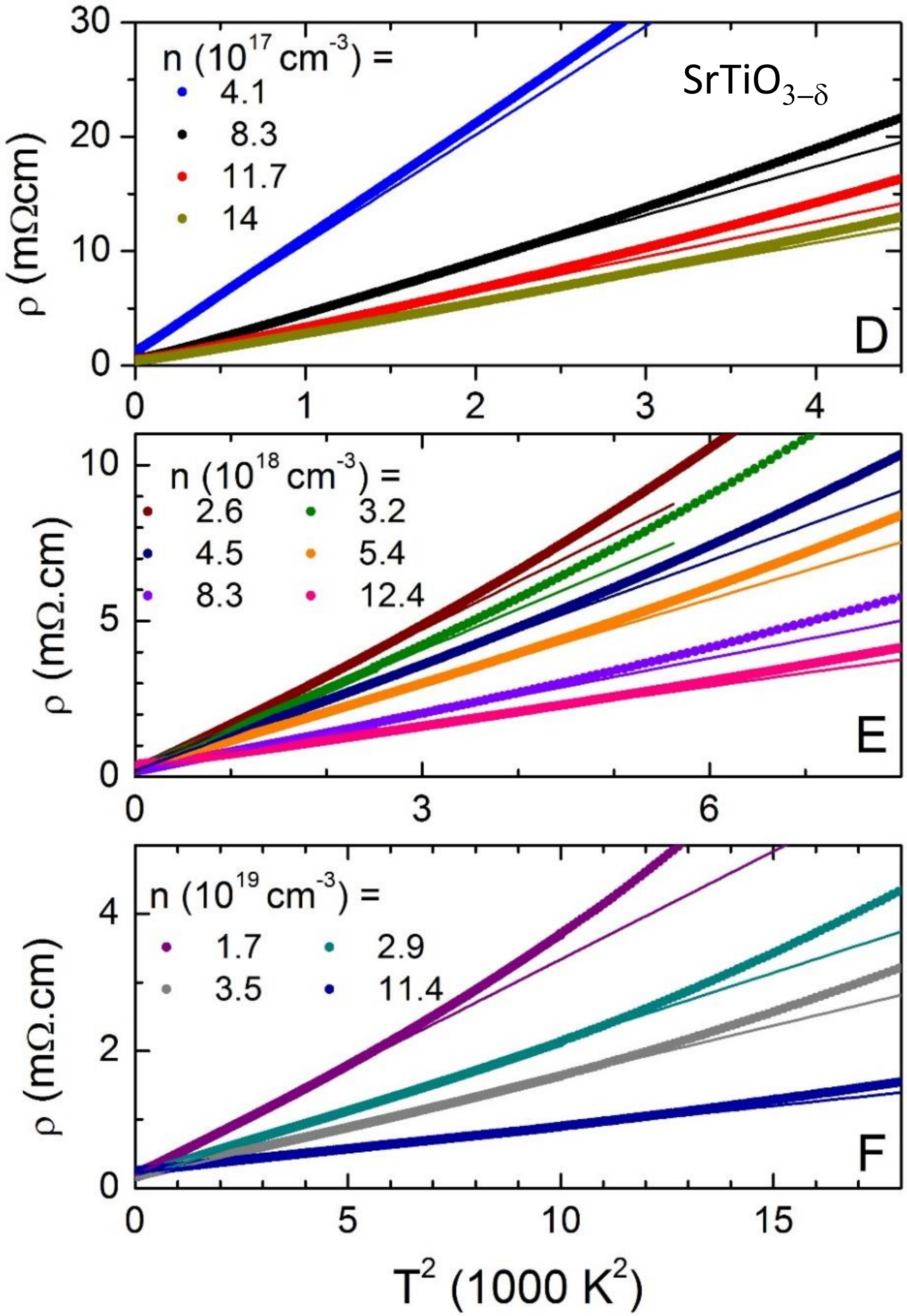
2015

Scalable T^2 resistivity in a small single-component Fermi surface

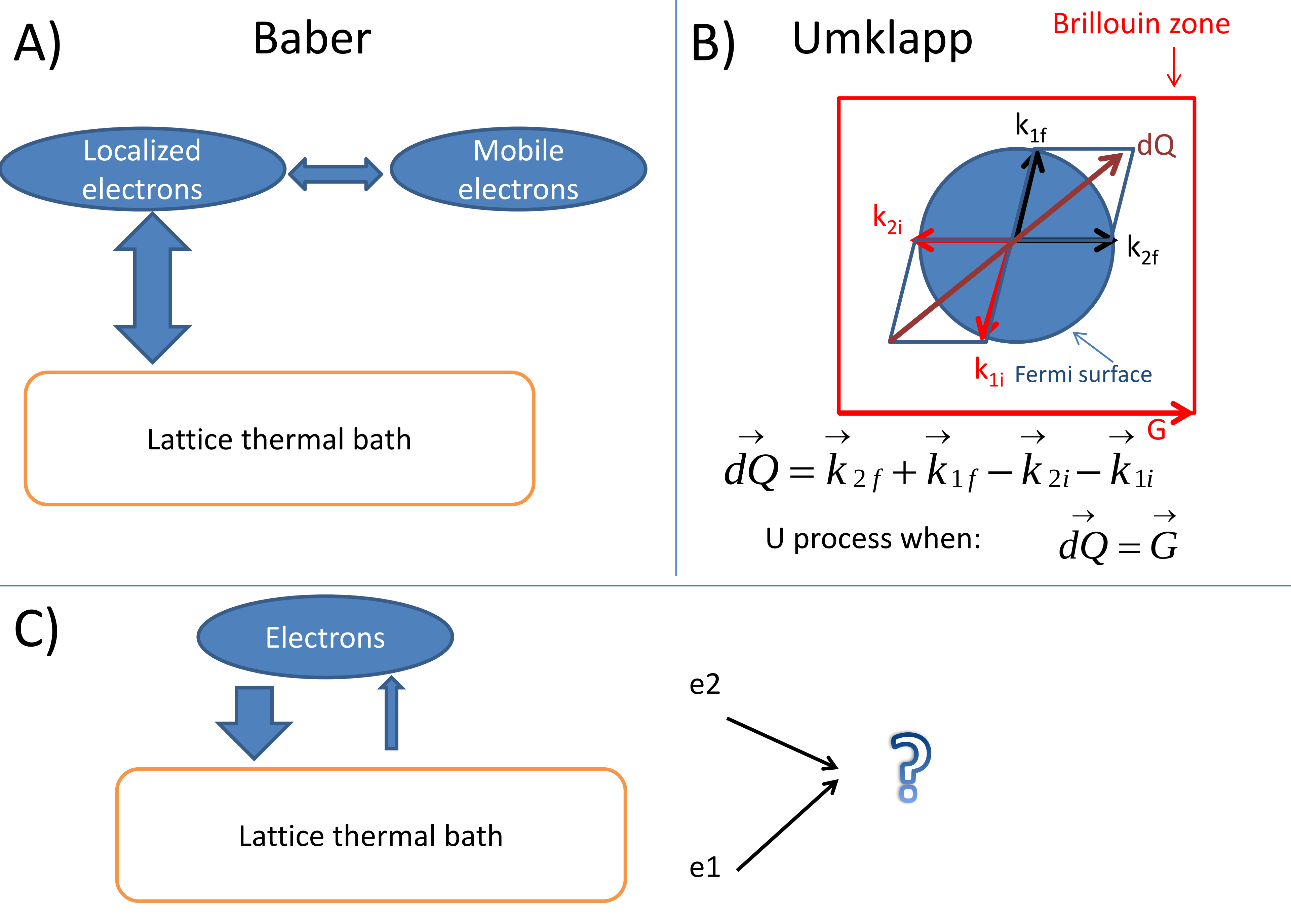
Xiao Lin, Benoît Fauqué, Kamran Behnia*

$$\rho = \rho_0 + AT^2$$





Non of the two mechanisms work!

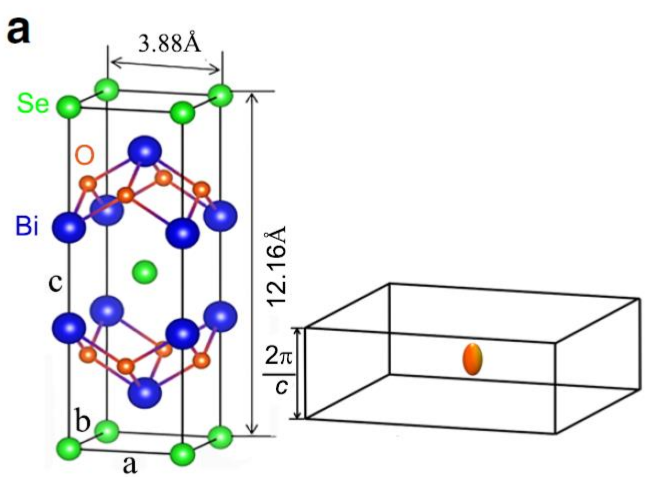

A)
Baber
Localized electrons
Mobile electrons
Lattice thermal bath
B)
Umklapp
Brillouin zone
k1f
dQ
k2i
k2f
k1i
Fermi surface
G
$\vec{dQ} = \vec{k}_{2f} + \vec{k}_{1f} - \vec{k}_{2i} - \vec{k}_{1i}$
U process when: $\vec{dQ} = \vec{G}$
C)
Electrons
Lattice thermal bath
e2
e1
?

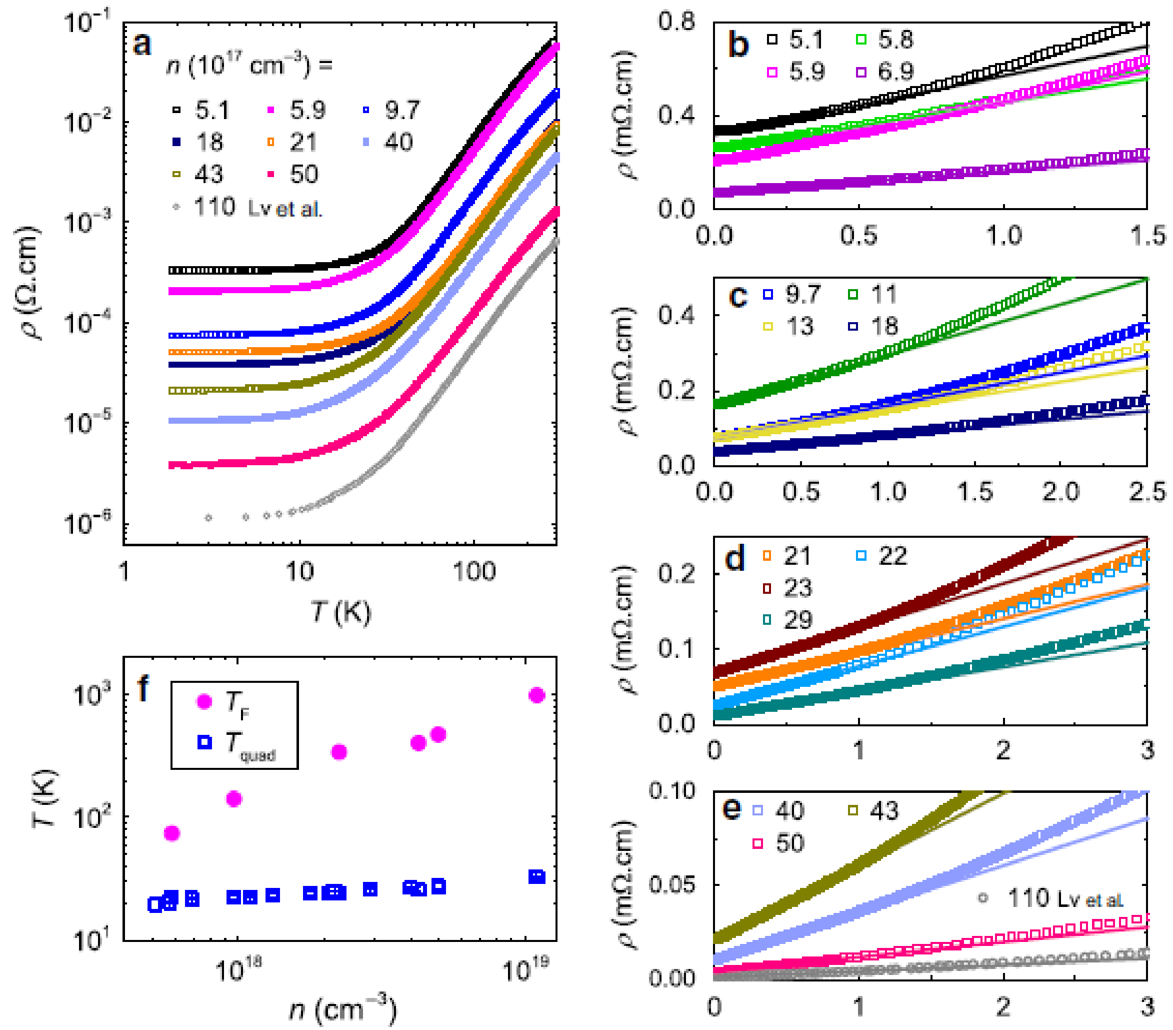
https://doi.org/10.1038/s41467-020-17692-6 OPEN

T-square resistivity without Umklapp scattering in dilute metallic Bi_2O_2Se

Jialu Wang[1,2], Jing Wu[1,2], Tao Wang[1,2], Zhuokai Xu[1,2], Jifeng Wu[1,2], Wanghua Hu[1,2], Zhi Ren[1,2], Shi Liu[1,2], Kamran Behnia[3] & Xiao Lin[1,2]✉

STO is not alone!





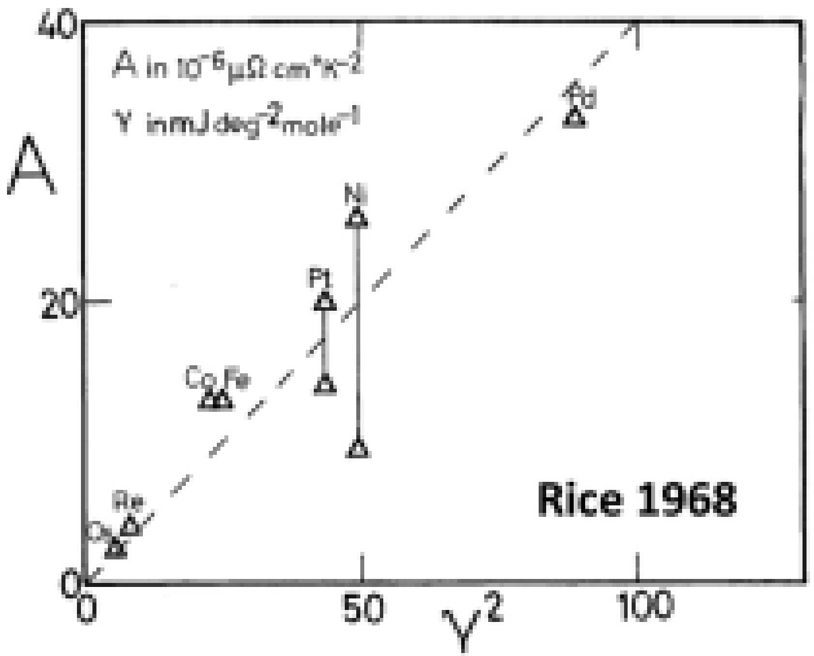
What sets the magnitude of A?

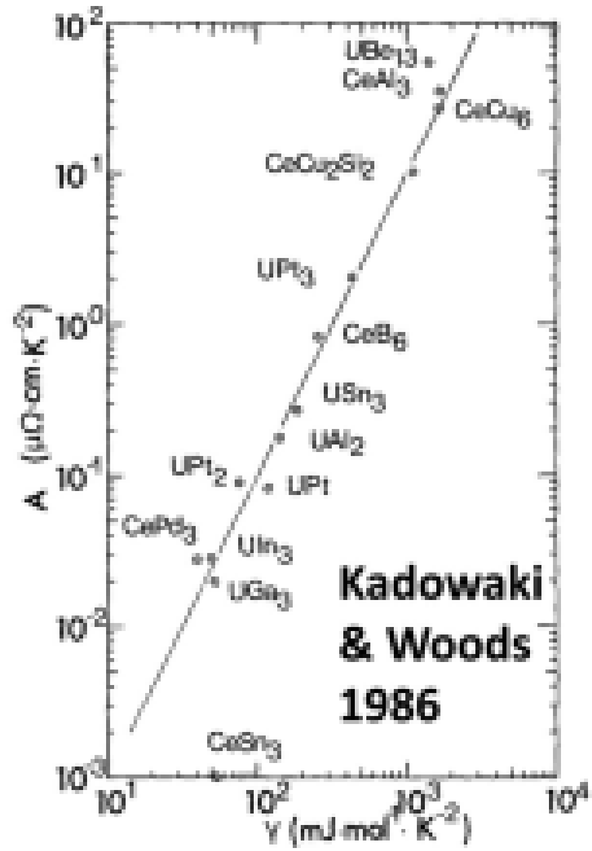
VOLUME 20, NUMBER 25 PHYSICAL REVIEW LETTERS 17 JUNE 1968

$$\rho = \rho_0 + AT^2$$

ELECTRON-ELECTRON SCATTERING IN TRANSITION METALS

M. J. Rice*
Solid State Theory Group, Department of Physics, Imperial College, London, England
(Received 15 April 1968)





$A \propto \gamma^2$ **γ :T-linear specific heat**

The KW scaling works only for **dense** metals

- Specific heat $\gamma = \frac{\pi^2}{2} k_B^2 \frac{n}{E_F}$ ← Number of électrons per unit cell

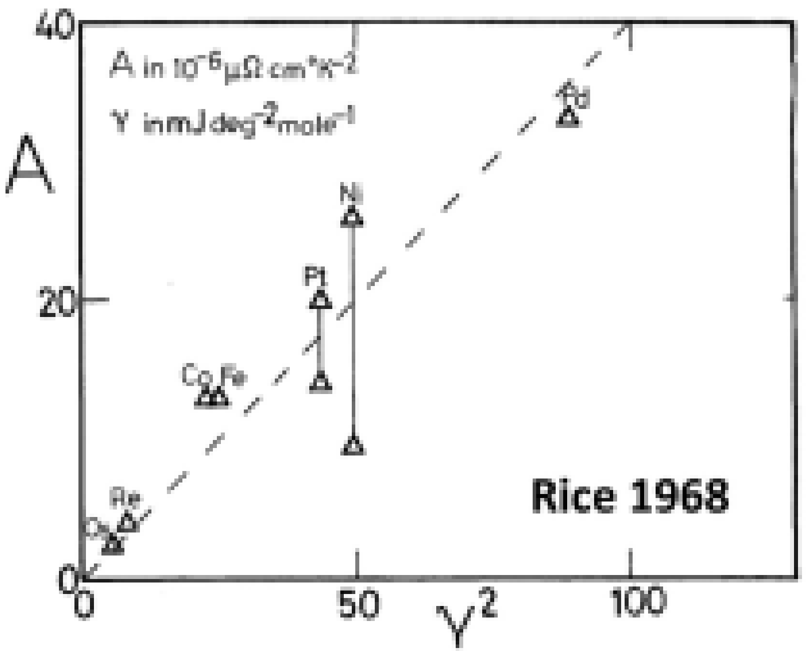
- T^2-resistivity $\rho = \rho_0 + AT^2$

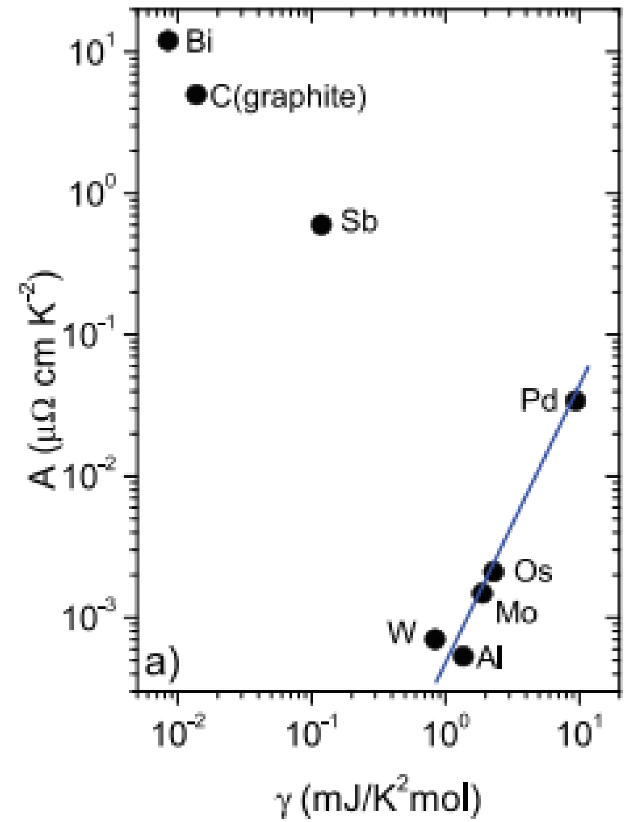
$$A = \frac{\hbar}{e^2}\left(\frac{k_B}{E_F}\right)^2 \ell_{quad}$$

n~1 ⇒ $A \propto \gamma^2$

Whatever n ⇒ $A \propto \frac{1}{E_F^2}$

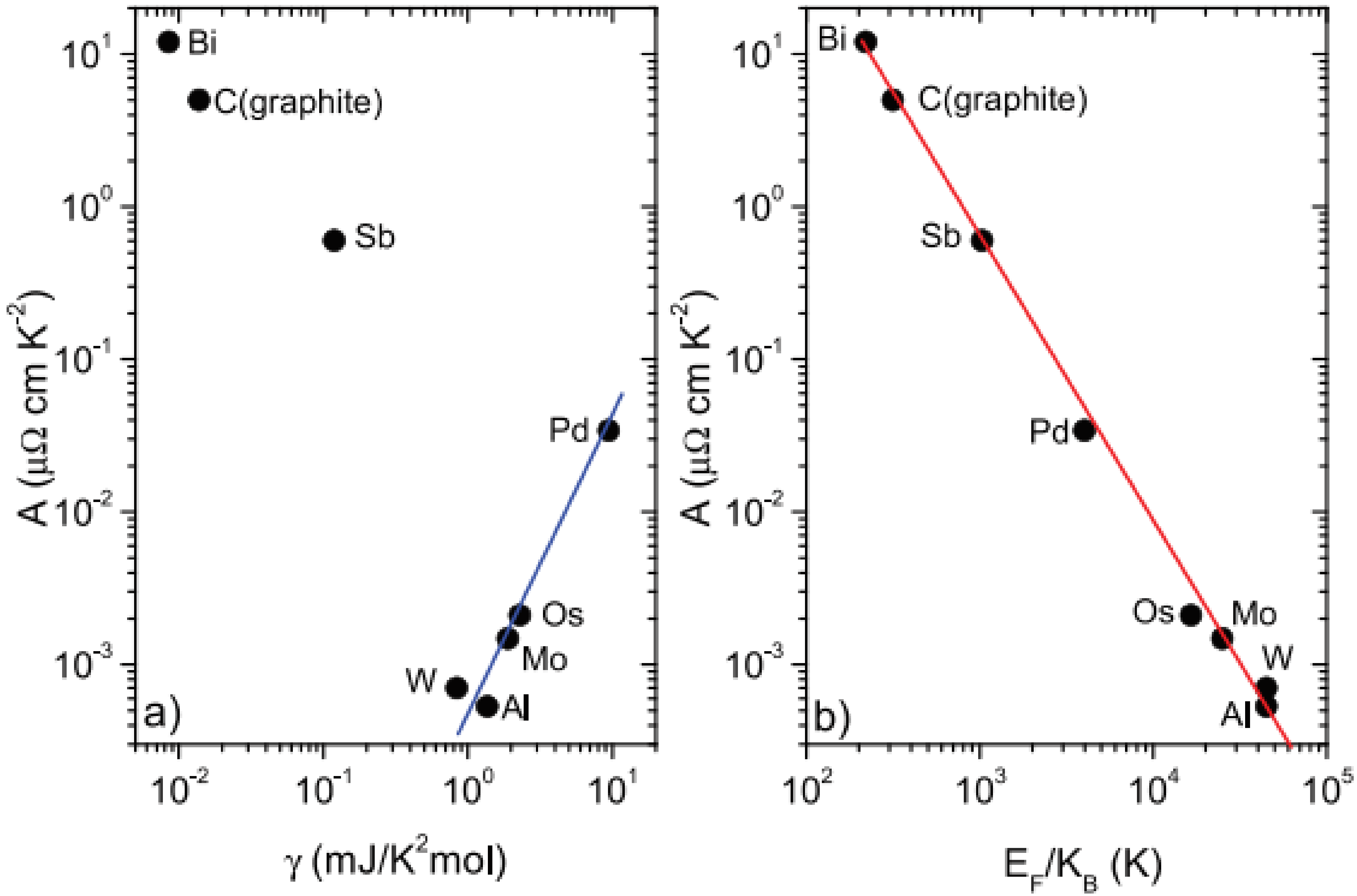
The dilute metals forgotten by Rice





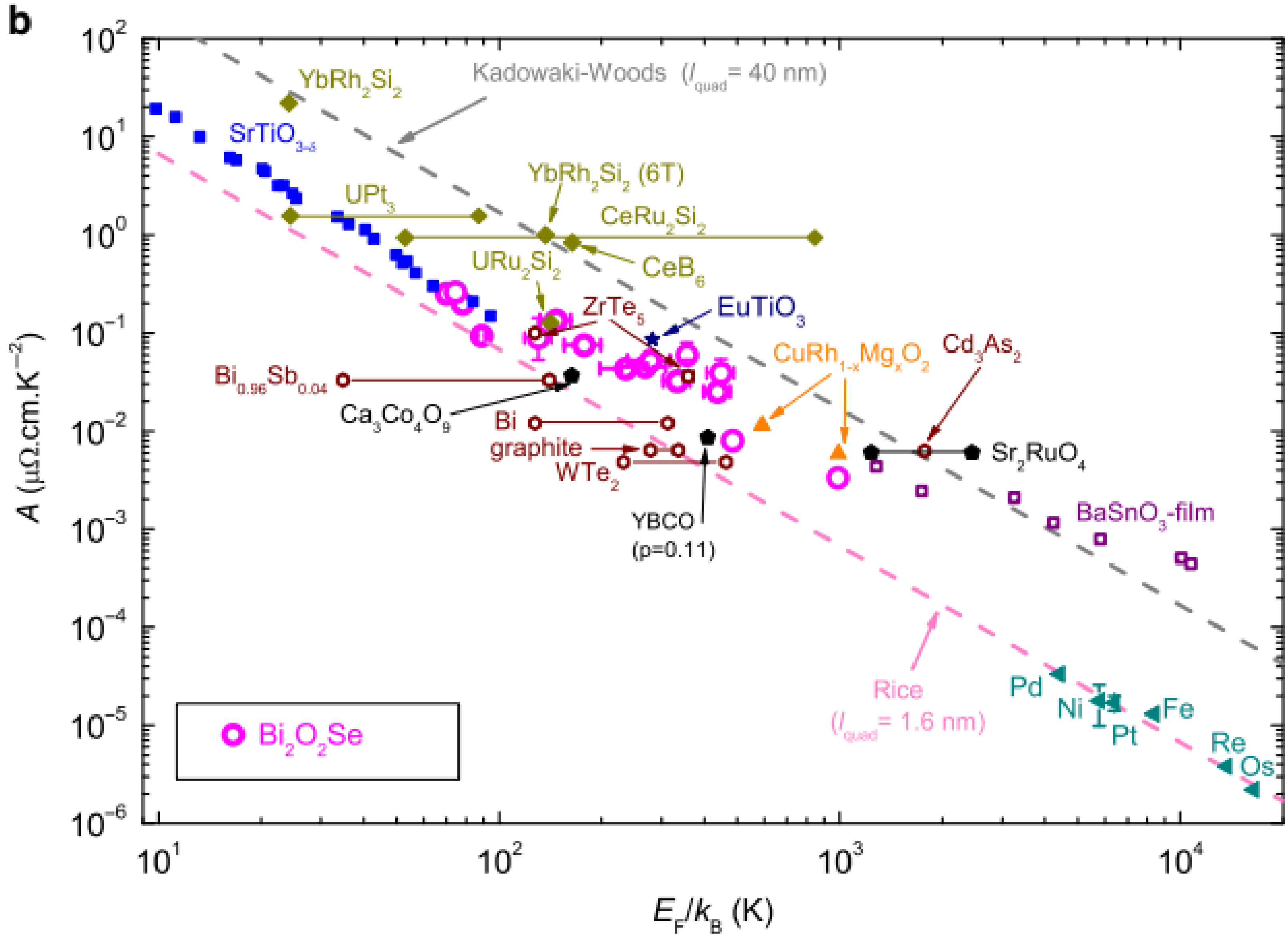
Element	Carrier density [m^3]	γ [mJ mol^{-1} K^{-2}]	E_F (K)	A [nΩ cm K^{-2}]
Bi	6×10^{23}	0.0085	220	12
C (graphite)	6×10^{24}	0.0138	315	5
Sb	1.1×10^{26}	0.119	1030	0.6
Mo	2.5×10^{28}	1.9	2.5×10^{4}	1.5×10^{-3}
W	1.4×10^{28}	0.84	4.5×10^{4}	7×10^{-4}
Pd	5×10^{28}	9.43	4×10^{3}	0.034
Al	6×10^{28}	1.37	4.5×10^{4}	5.3×10^{4}

The "extended Kadowaki-Woods" scaling



Knowing the Fermi Energy, one can predict the magnitude of A.

The “extended Kadowaki-Woods” scaling



Knowing the Fermi Energy, one can predict the magnitude of A.

2 questions about the origin of T-square resistivity in Fermi liquids

- Why is it universally linked to the Fermi energy?
- Why does it persist without Umklapp?

Let us turn our attention to thermal transport.

The law of Wiedemann and Franz

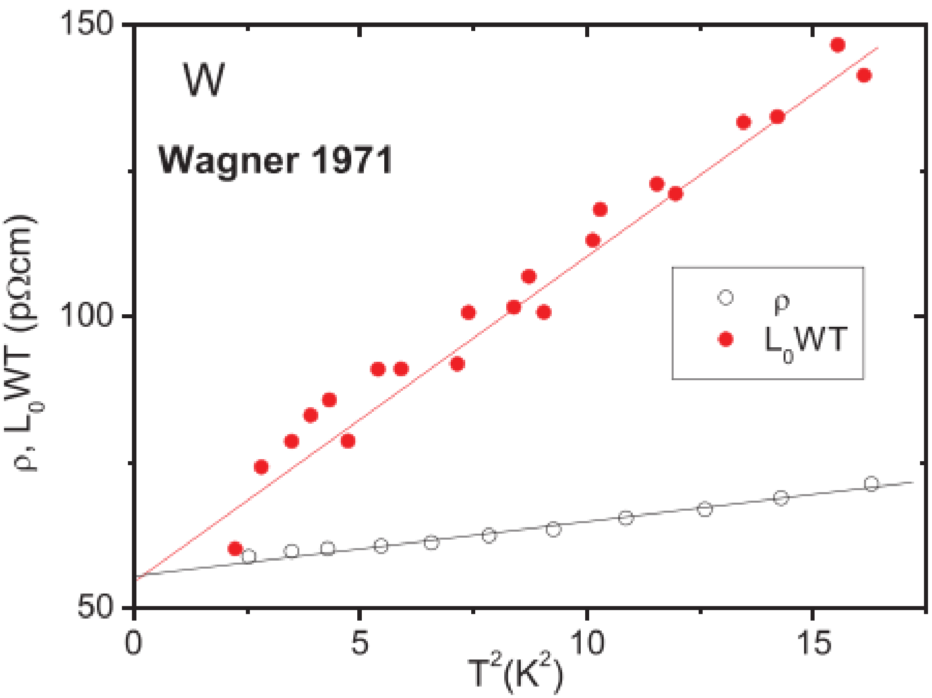
Wiedemann G and Franz R 1853 Ann. Phys., Lpz. **89** (2) 497

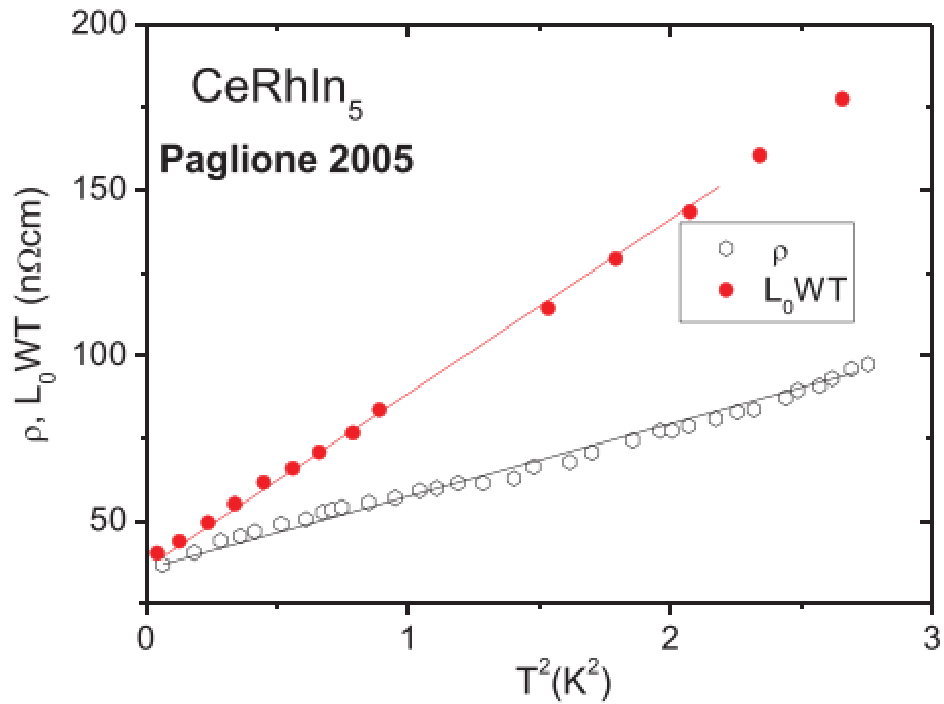
$$L = \frac{\kappa}{T\sigma}$$ Is the same in different metals

Table 3 Experimental Lorenz numbers

Metal	$L \times 10^8$ watt-ohms/deg^2 0°C	100°C	Metal	$L \times 10^8$ watt-ohms/deg^2 0°C	100°C
Ag	2.31	2.37	Pb	2.47	2.56
Au	2.35	2.40	Pt	2.51	2.60
Cd	2.42	2.43	Sn	2.52	2.49
Cu	2.23	2.33	W	3.04	3.20
Ir	2.49	2.49	Zn	2.31	2.33
Mo	2.61	2.79			

T-square thermal resistivity





$$L_0 = \frac{\pi^2}{3}\left(\frac{\kappa_B}{e}\right)^2$$

$$WT = (WT)_0 + BT^2$$

$$\rho_0 = L_0(WT)_0$$

$L_0 B$ >A

$$WT = \left(\frac{\kappa}{T}\right)^{-1}$$

$$\rho = \rho_0 + AT^2$$

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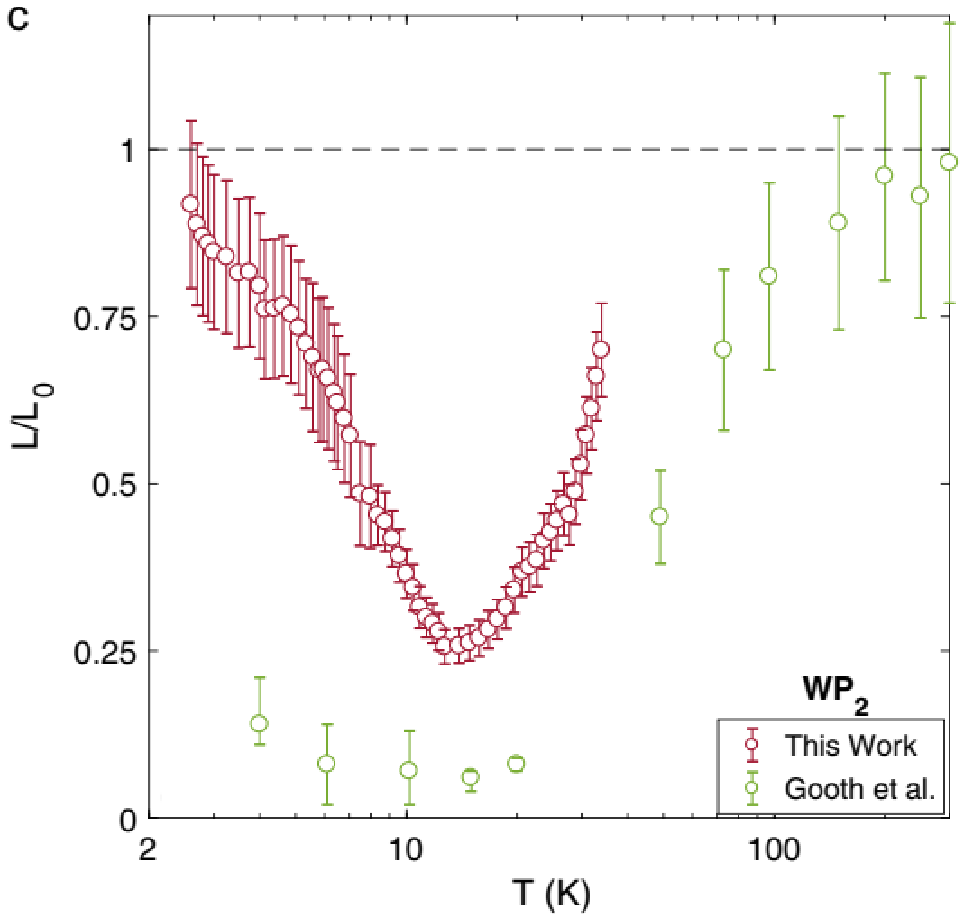
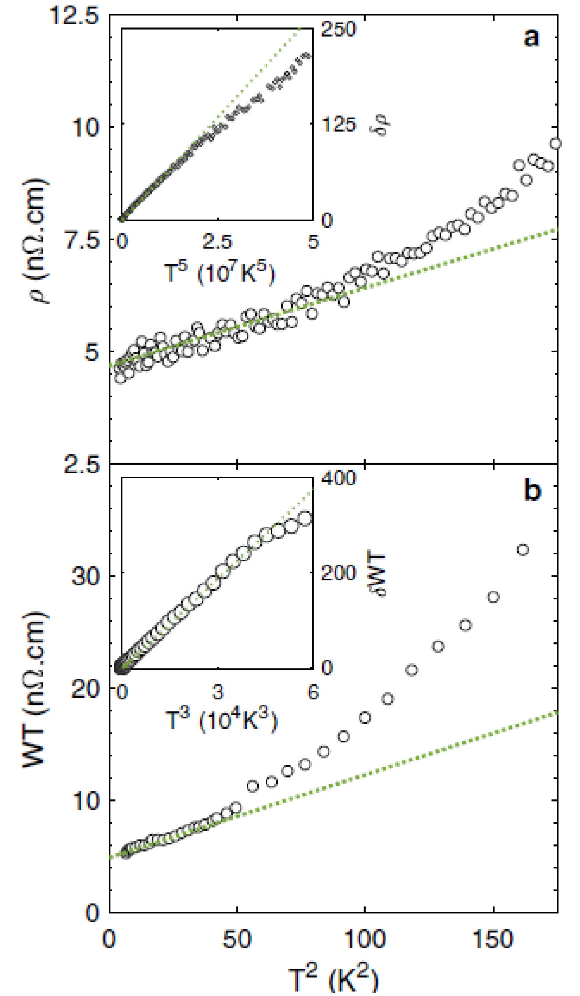
ARTICLE OPEN

Departure from the Wiedemann–Franz law in WP_2 driven by mismatch in *T*-square resistivity prefactors

Alexandre Jaoui[1,2], Benoît Fauqué[1,2], Carl Willem Rischau[2,3], Alaska Subedi[4,5], Chenguang Fu[6], Johannes Gooth[6], Nitesh Kumar[6], Vicky Süß[6], Dmitrii L. Maslov[7], Claudia Felser[6] and Kamran Behnia[2,8]

T (K)





$$\rho = \rho_0 + A_2 T^2 + A_5 T^5$$

$$WT = W_0 T + B_2 T^2 + B_3 T^3$$

$B_2 / A_2 \sim 5$

What sets the mismatch between the two T-square prefactors in a given solid?

Material	ρ_0 (nΩcm)	A_2 (pΩcmK^{-2})	B_2(pΩcmK^{-2})	B_2/A_2
WP_2	4	17	76	5
W	0.06	0.9	6.2	6
Ni	3	25	61	2.5
UPt_3	200	$1.6\ 10^6$	$2.4\ 10^6$	1.5
$CeRhIn_5$	37	$2.1\ 10^4$	$5.7\ 10^4$	2.5

Theory:

- Herring (1967): The ratio is quasi-universal ~ 2!
- Li & Maslov (2019) : No boundary! It can become arbitrarily large!

Does T-square thermal resistivity require Umklapp events?

The other fermion…

What happens to Liquid Helium 3 at very low Temperatures?

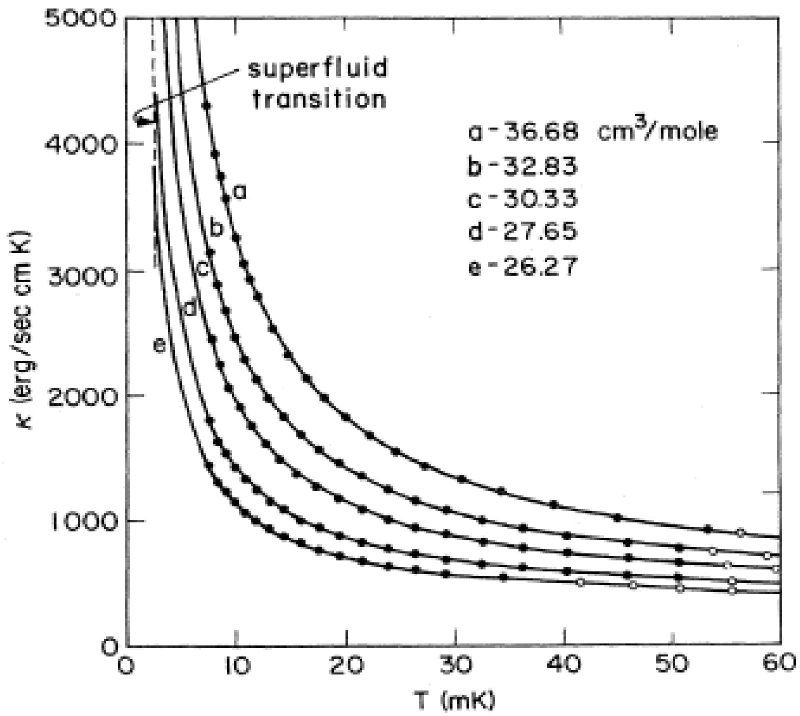
By E. R. Dobbs, London*)

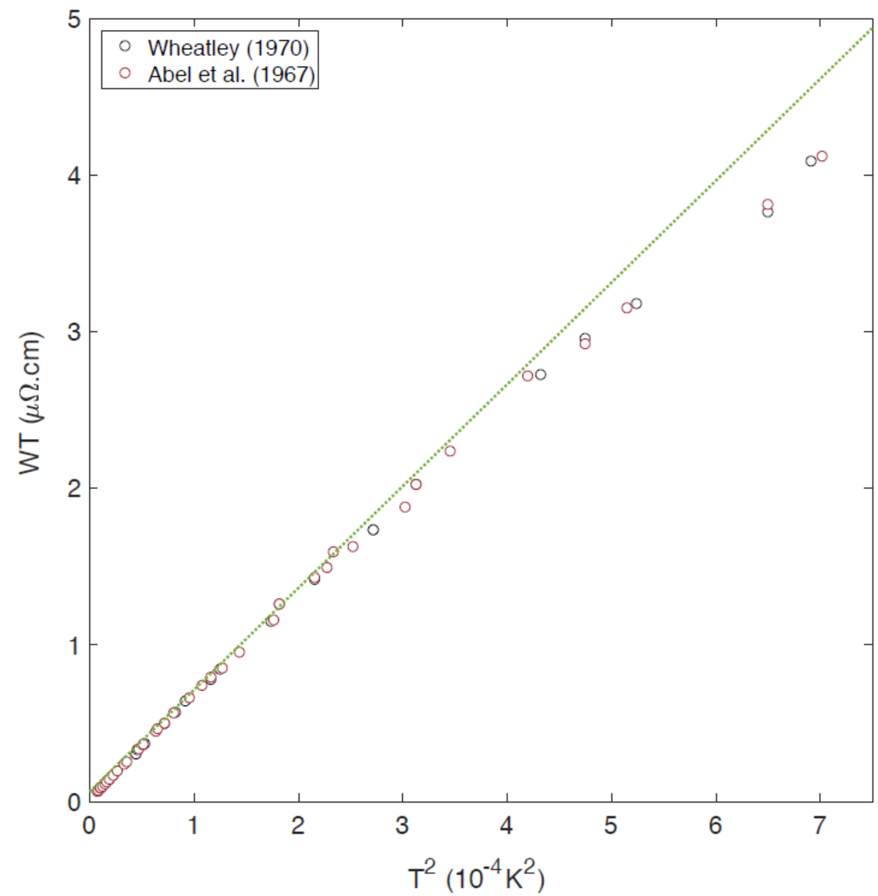
The time between collisions is proportional to ***T*$^{-2}$…**, the viscosity of ^{3}He rises dramatically…, becoming at **3** mK, the same as olive oil at **40 °C!**

PHYSICAL REVIEW B VOLUME 29, NUMBER 9 1 MAY 1984

Thermal conductivity of normal liquid ^{3}He

Dennis S. Greywall
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
(Received 13 October 1983)





- In ^{3}He: $\kappa \propto T^{-1}$
- $WT \equiv (\frac{\kappa}{T})^{-1} \propto T^2$

THE THEORY OF A FERMI LIQUID

(The Properties of Liquid ^{3}He at Low Temperatures)

By A. A. ABRIKOSOV and I. M. KHALATNIKOV
Institute for Physical Problems, Moscow

Viscosity:

$$\eta = \frac{64}{45} T^{-2} \frac{\hbar^3 p_0^5}{m^{*4}} \left\{ \left[\frac{w(\theta,\phi)}{\cos\frac{1}{2}\theta} (1-\cos\theta)^2 \sin^2\phi \right]_{av} \right\}^{-1}$$

Thermal conductivity:

$$\kappa = \frac{8}{3} \frac{\pi^2 \hbar^3 p_0^3}{m^{*4} T} \left\{ \left[\frac{w(\theta,\phi)(1-\cos\theta)}{\cos\frac{1}{2}\theta} \right]_{av} \right\}^{-1}.$$

Origin of T-square thermal resistivity in ^{3}He

$$D = \frac{1}{3}\tau v_m^2$$

Diffusivity

Scattering time

Mean velocity

$$\tau \propto T^{-2}$$

Energy diffusivity: $D \propto T^{-2}$

Specific heat: $C \propto T$

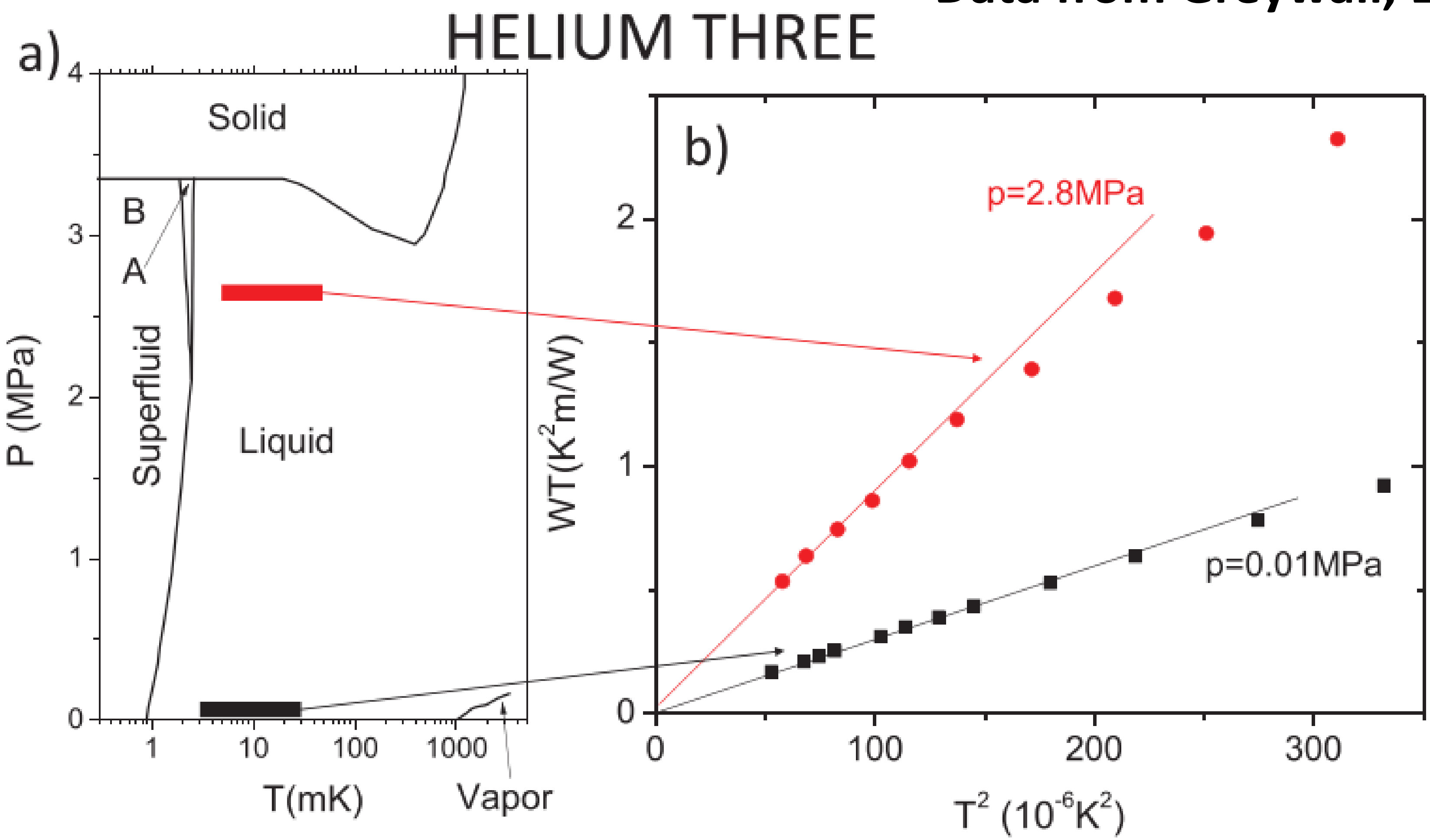
Momentum diffusivity (Viscosity): $\eta \propto T^{-2}$

$\kappa = C \times D \propto T^{-1}$

$$WT \propto T^2$$

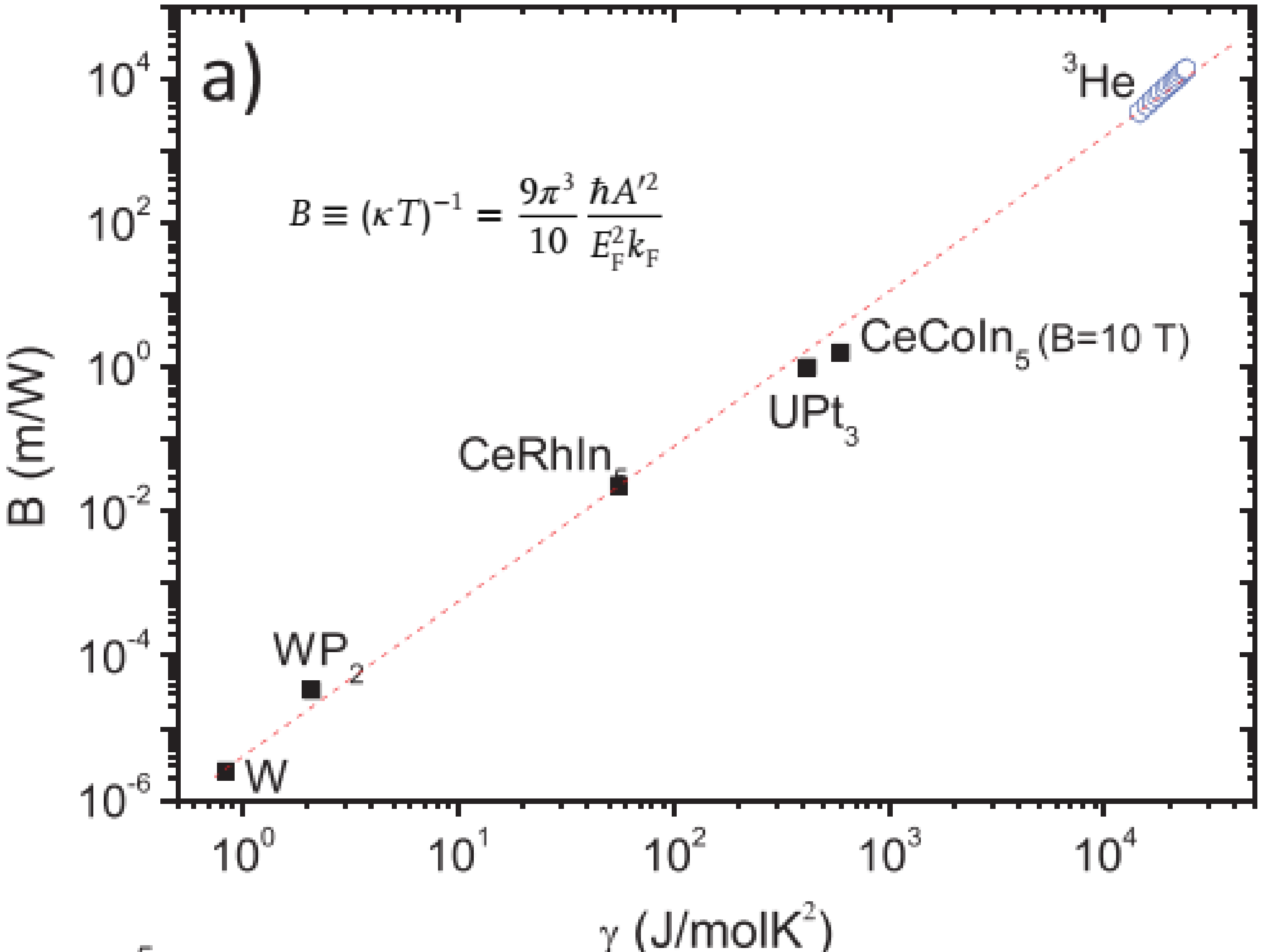
T-square thermal resistivity in ^{3}He

Data from Greywall, 1984



No Umklapp here!

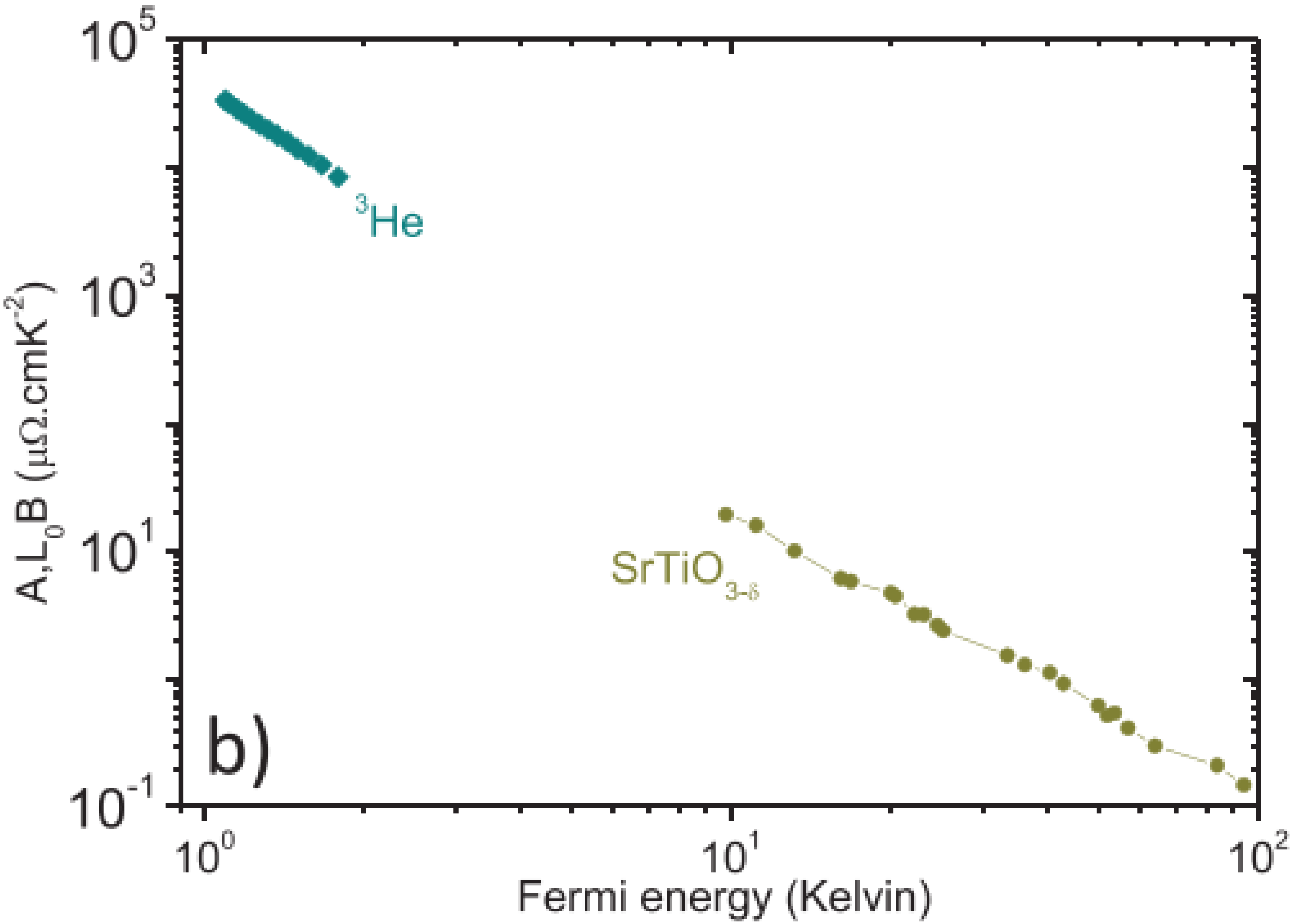
^{3}He and metals: a common thread



5

T-square thermal resistivity prefactors

^{3}He and metals



T-square electrical resistivity can occur without Umklapp

Two explanations of the T-square mismatch between electrical and thermal channels

- The electrical T-square prefactor (A) is NOT affected by horizontal events.
- The thermal T-square prefactor (B) is affected by both horizontal and vertical events.
- B>A, because some collisions are horizontal!

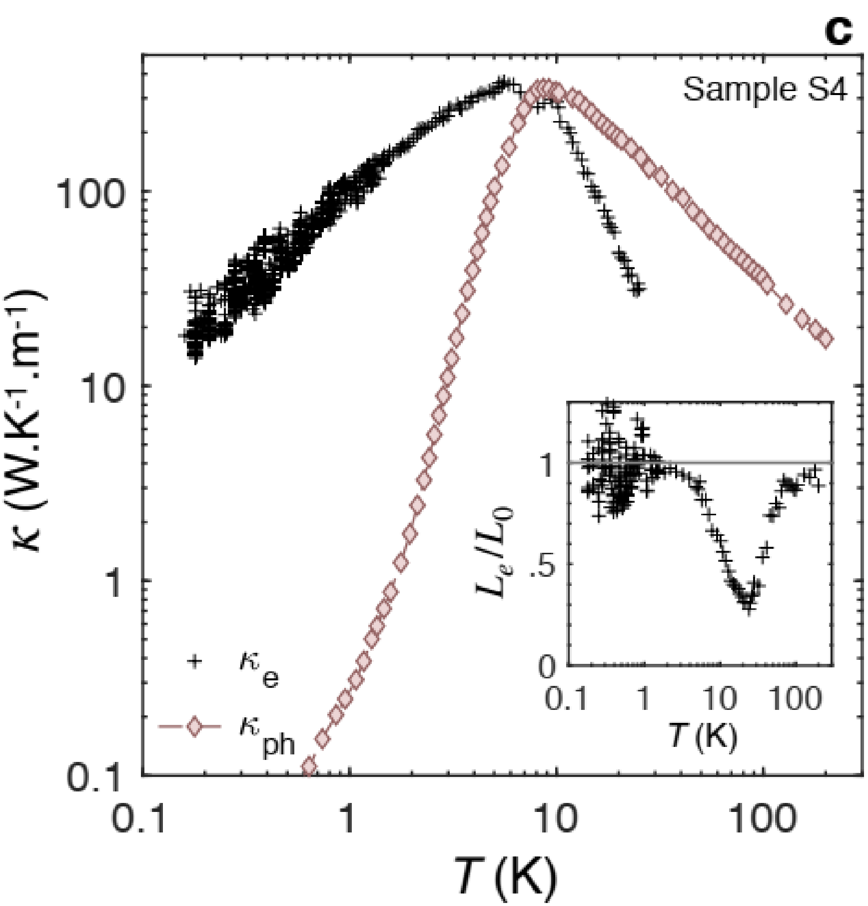
- The electrical T-square prefactor (A) quantifies momentum-relaxing collisions.
- The thermal T-square prefactor (B) quantifies momentum-conserving collisions.
- B>A, because some e-e collisions conserve momentum!

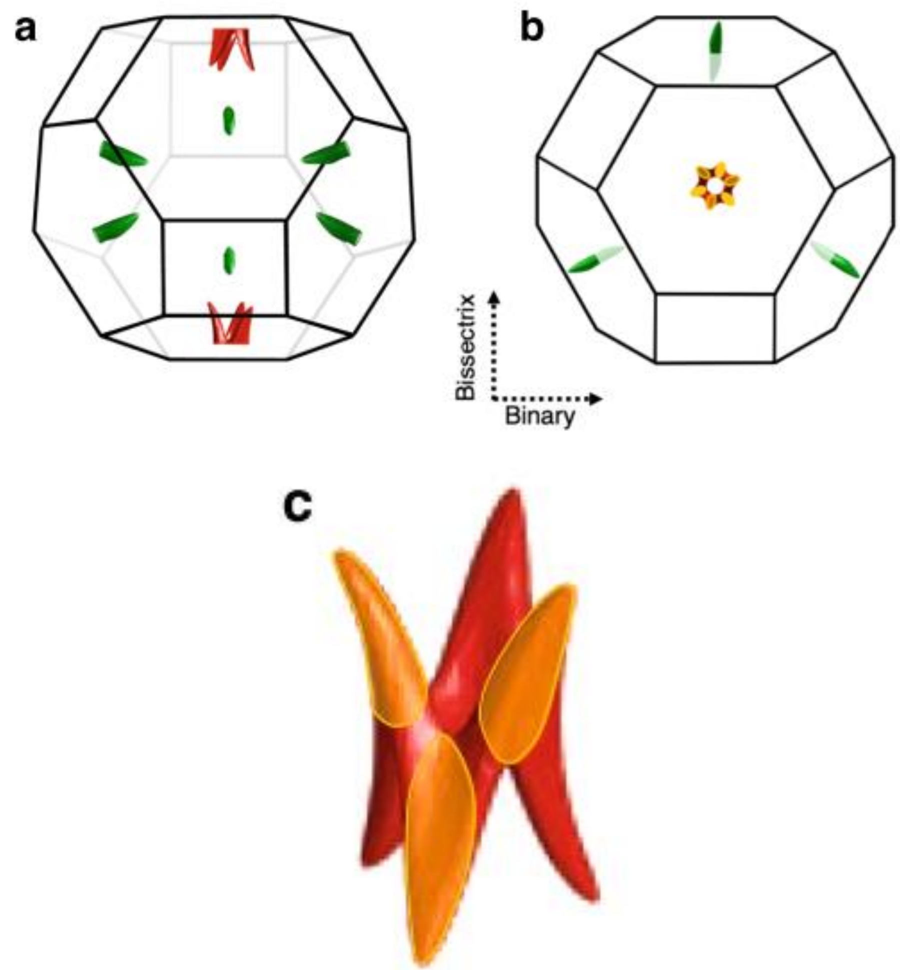
Look at the size dependence of B/A in a solid with ballistic electronic transport!

ARTICLE
https://doi.org/10.1038/s41467-020-20420-9 OPEN

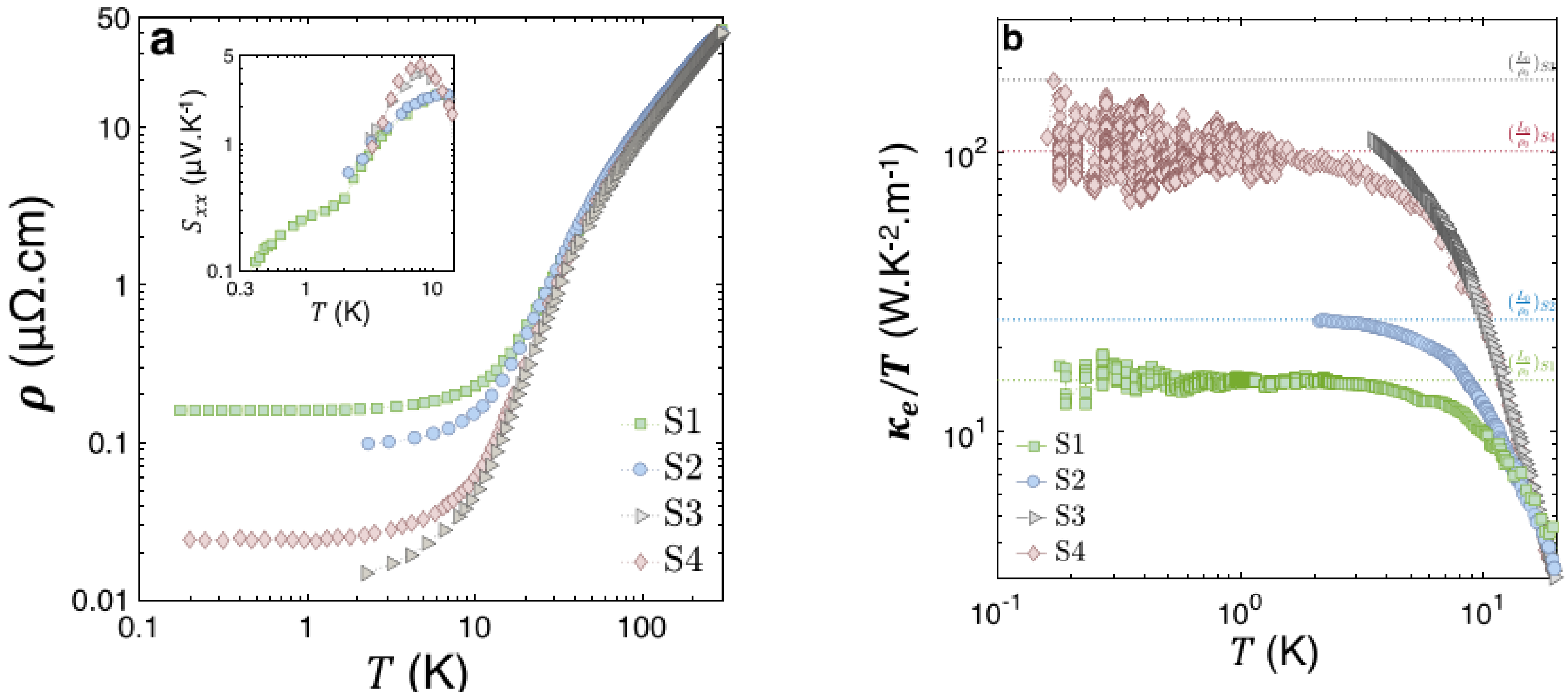
Thermal resistivity and hydrodynamics of the degenerate electron fluid in antimony

Alexandre Jaoui[1,2] ✉, Benoît Fauqué[1] & Kamran Behnia[2]

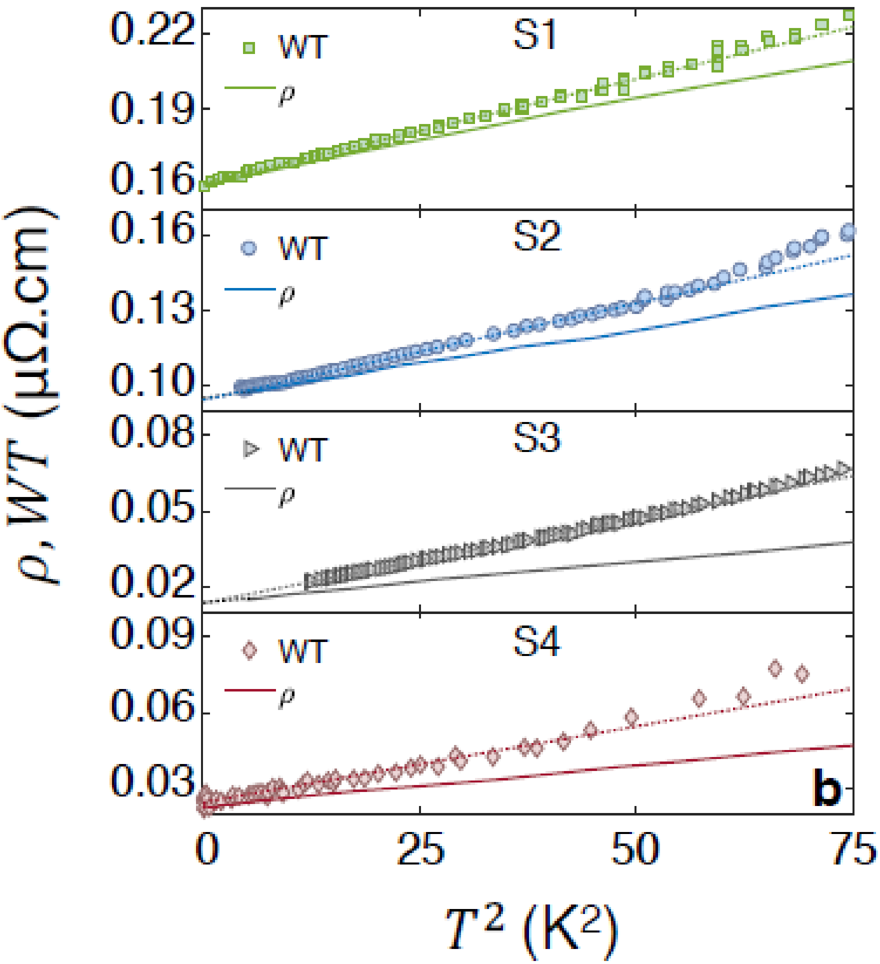




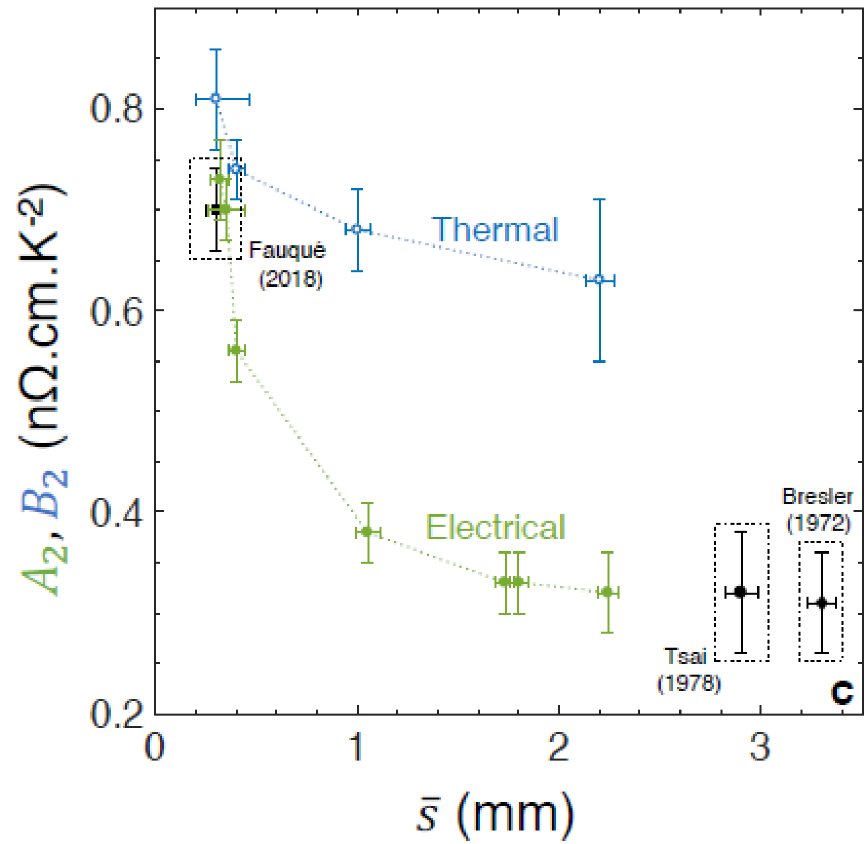
Electric conductivity and electronic thermal conductivities are both size dependent.



Evolution of T-square resistivities



Jaoui et al. (2021)



The larger the sample the higher the B/A ratio!

A fraction of e-e scattering is momentum-conserving

PRL **115,** 056603 (2015) PHYSICAL REVIEW LETTERS week ending 31 JULY 2015

Violation of the Wiedemann-Franz Law in Hydrodynamic Electron Liquids

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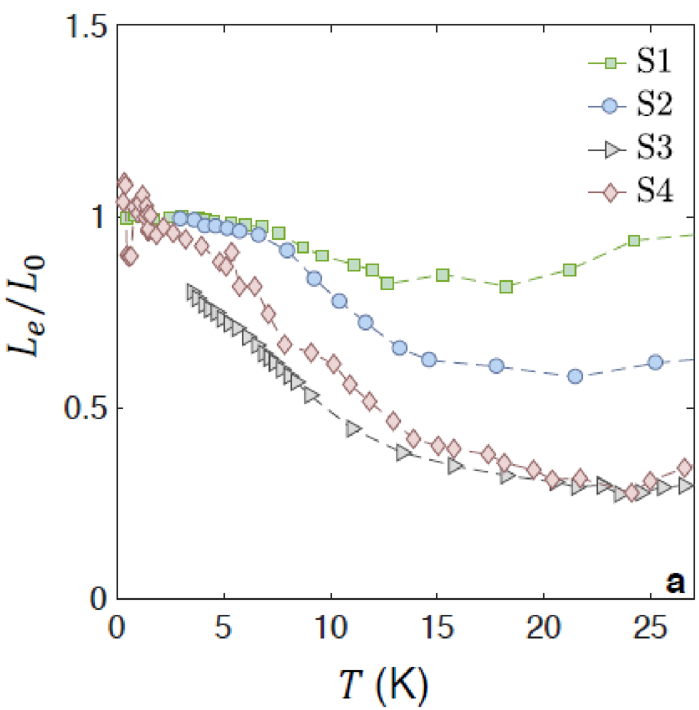
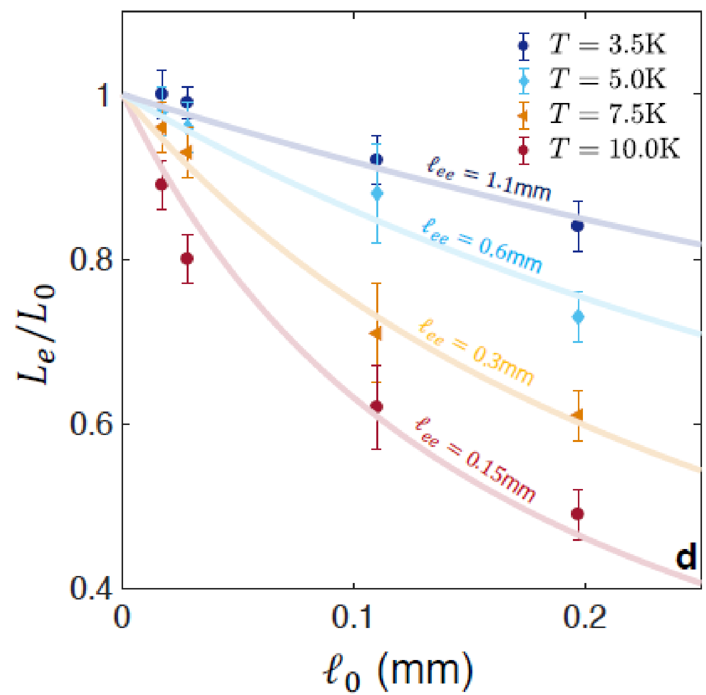
Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA
(Received 16 June 2014; revised manuscript received 16 January 2015; published 31 July 2015)





Jaoui et al. (2021)

phonons + electrons

PHYSICAL REVIEW X **12,** 031023 (2022)

Formation of an Electron-Phonon Bifluid in Bulk Antimony

Alexandre Jaoui,[1,2,*,†] Adrien Gourgout,[2] Gabriel Seyfarth,[3] Alaska Subedi,[4,5] Thomas Lorenz,[6]
Benoît Fauqué,[1] and Kamran Behnia[2,‡]

In elemental antimony

Phonons collide more frequently with electrons than with other phonons.

The flow between the two reservoirs is asymmetric: Phonon-phonon collisions conserve momentum whereas electron-electron collisions not (U e-e events).

- Phonons do not become ballistic (in contrast to électrons).
- Phonons display quantum oscillations.
- Electrons do not display T^5 resistivity.
- The Dingle mobility is decoupled from transport mobility.

The standard picture is modified by frequent e-ph collisions

Decreasing temperature

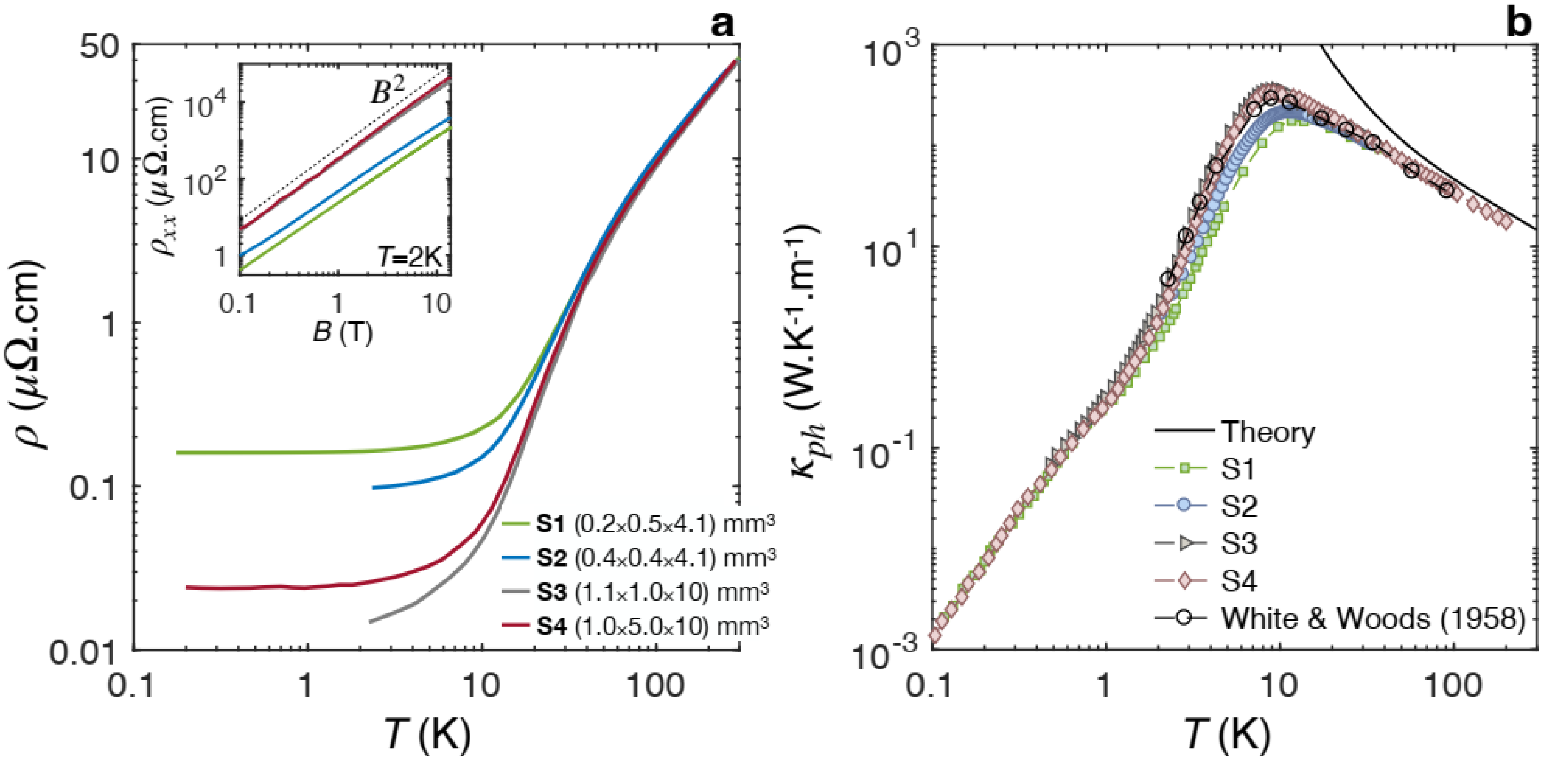
Phonon conductivity

- Ph-Ph scattering
  $\kappa \propto T^{-1}$
- Scattering by defects
  Peak in κ
- Scatt… by bound… es $\kappa \propto T^3$

Electron conductivity

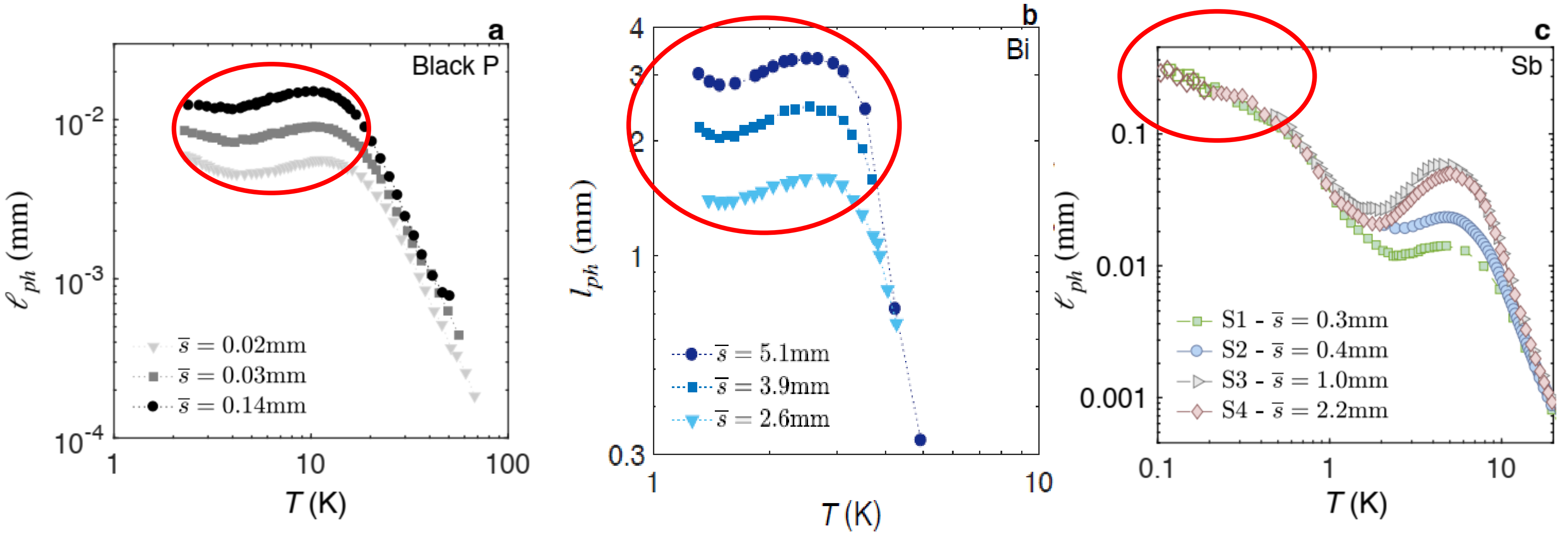
- Scattering by phonons
  $\rho \propto T$
- Scatte… by small-q phono… $\propto T^5$
- e-e scattering
  $\rho \propto T^2$
- Scattering by defects and boundaries ρ_0

In Sb electrons become quasi-ballistic, but non phonons



Why?

A comparison of three solids



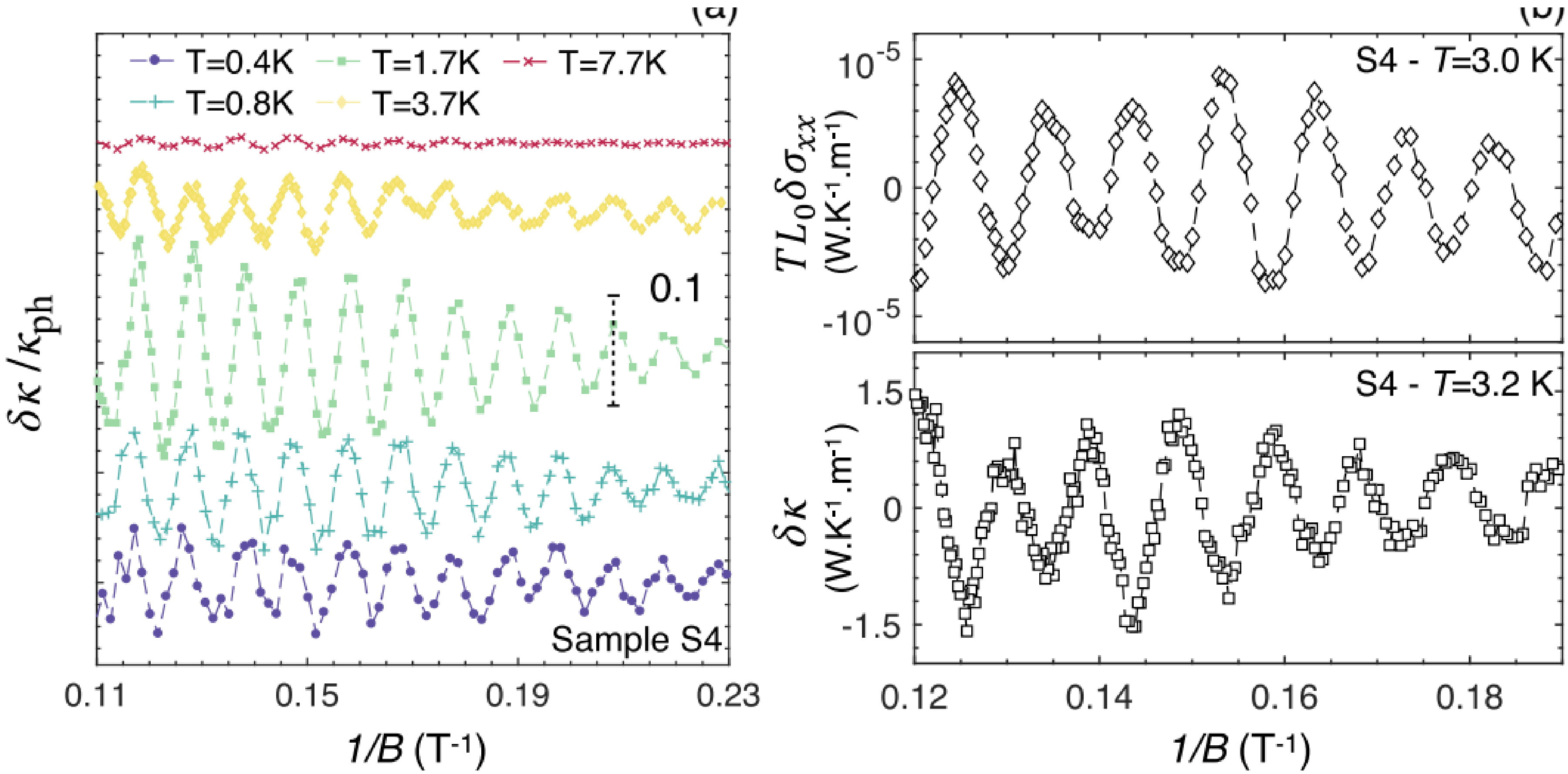
Insulator

$n=p=3X10^{17}$ cm^{-3}

$n=p=6X10^{18}$cm^{-3}

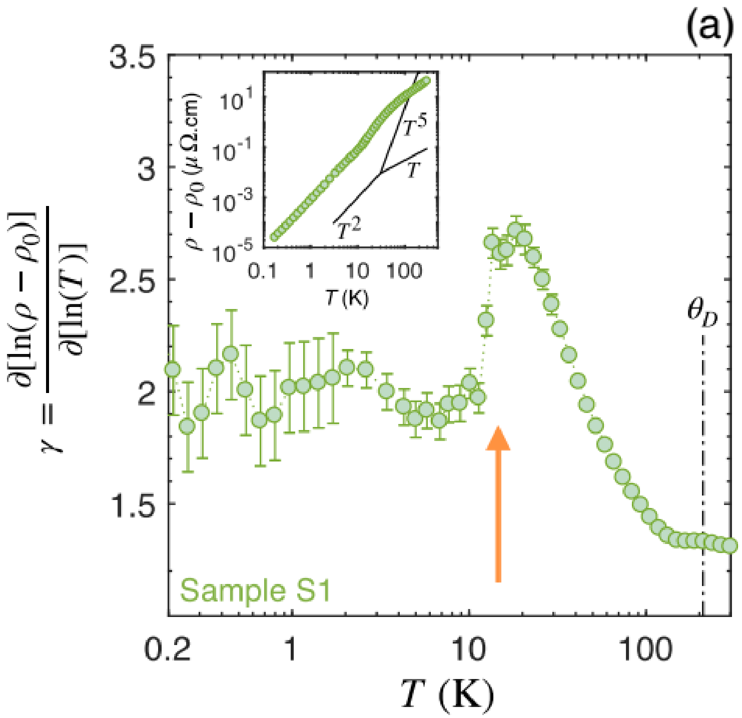
- Acoustic phonons are scattered by long wavelength electrons in Sb down to 0.1 K!
- A consequence of quasi-commensurability between the wavelength of electrons and phonons!

Quantum oscillations of thermal conductivity in Sb



Phonon thermal conductivity enhances each time a Landau level is evacuated.

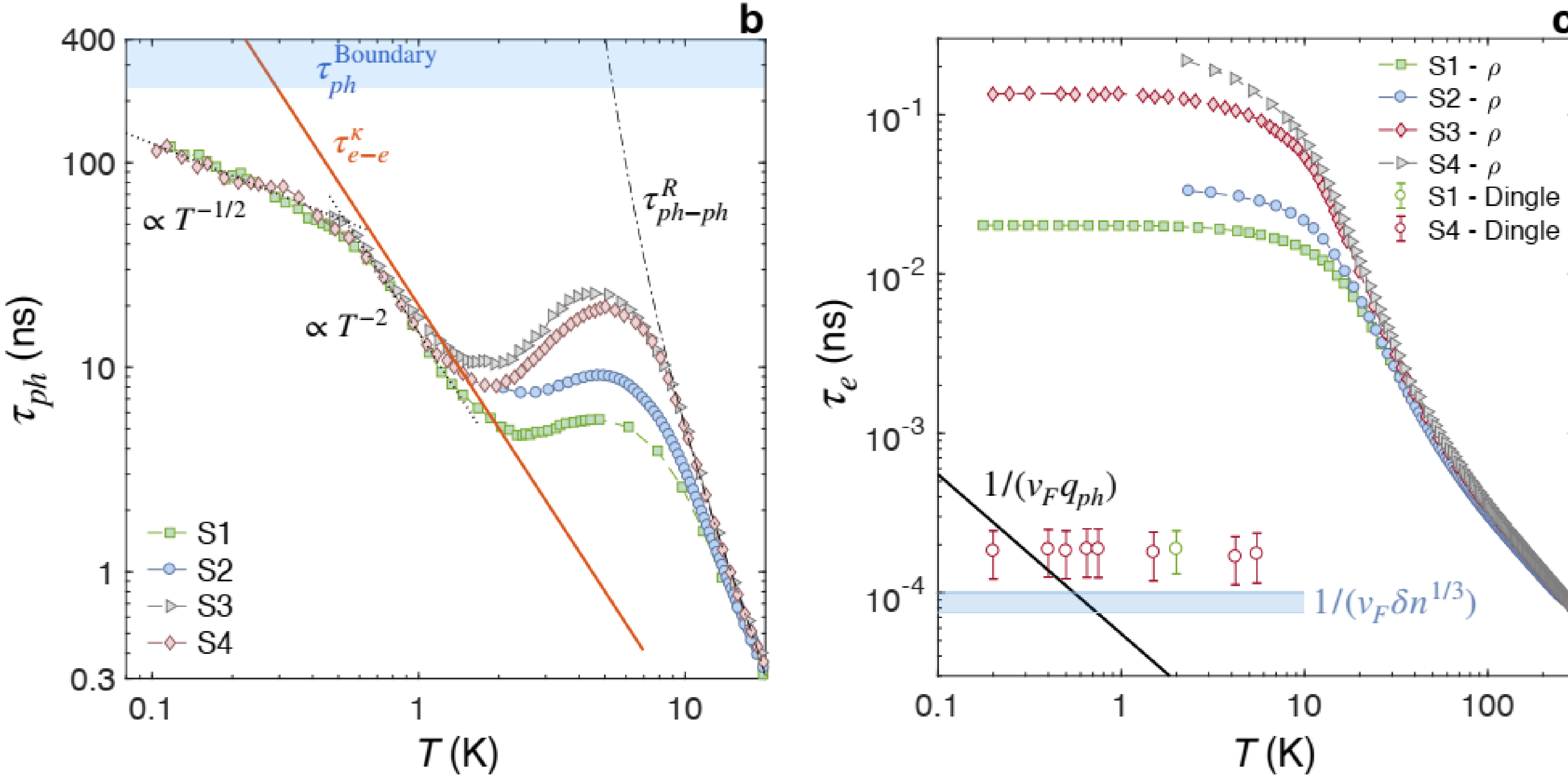
Absence of T^5 resistivity at low T



$$\rho = \rho_0 + T^\gamma$$

- For e-ph scattering, one expects $\gamma = 5, when\ T \ll \Theta_D$. Here $\gamma = 2$
- Scattering by phonons does not decay the charge current!
- The only possibility to inelastically loose momentum for electrons is scattering by other electrons.

Time scales



Electron-electron scattering is partially about exchanging phonons!

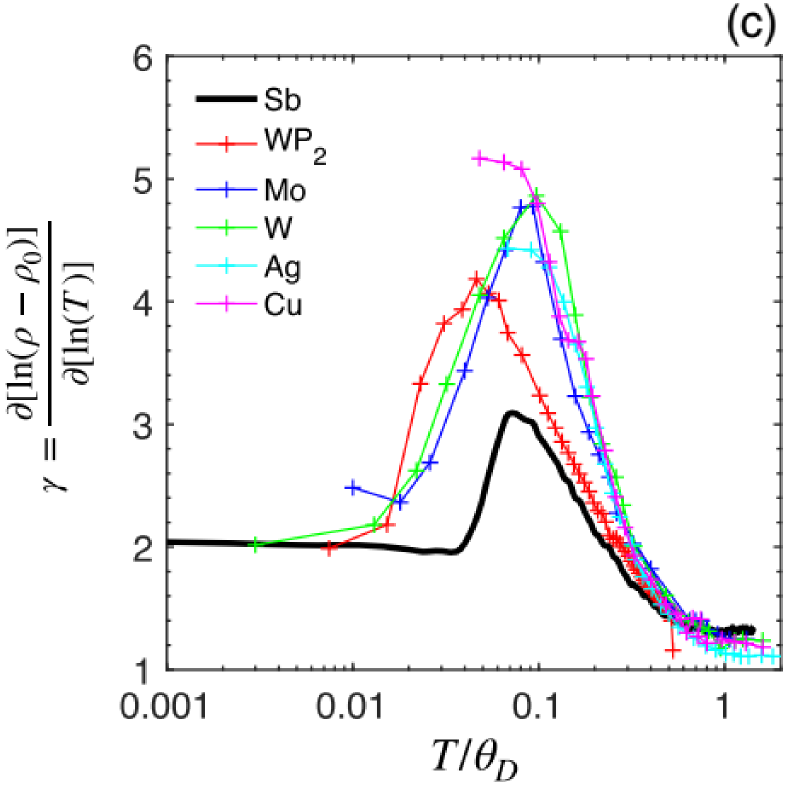
Summary

- In some solids, in a finite temperature window, phonon flow is amplified by momentum-conserving collisions. This is the Gurzhi's hydrodynamic regime.

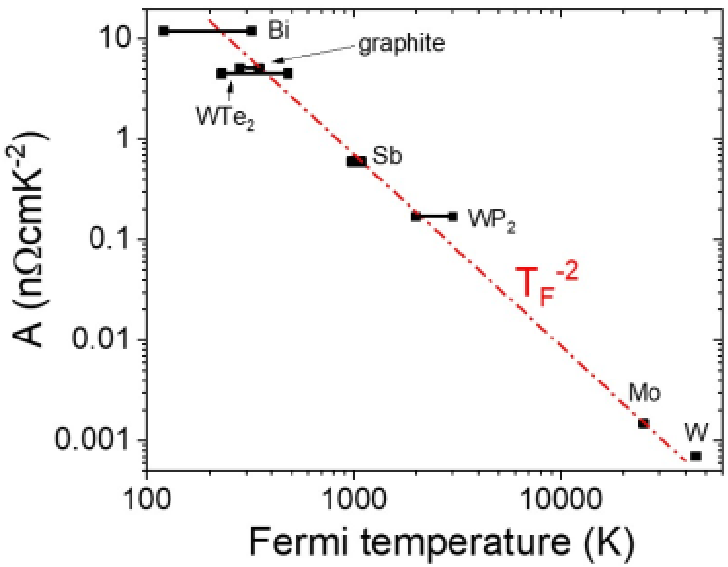
- T-square thermal resistivity in metals and in ^{3}He scale together.

- In macroscopic crystals of antimony an electron-phonon bifluid emeges at cryogenic temperattures.

Comparison with other metals

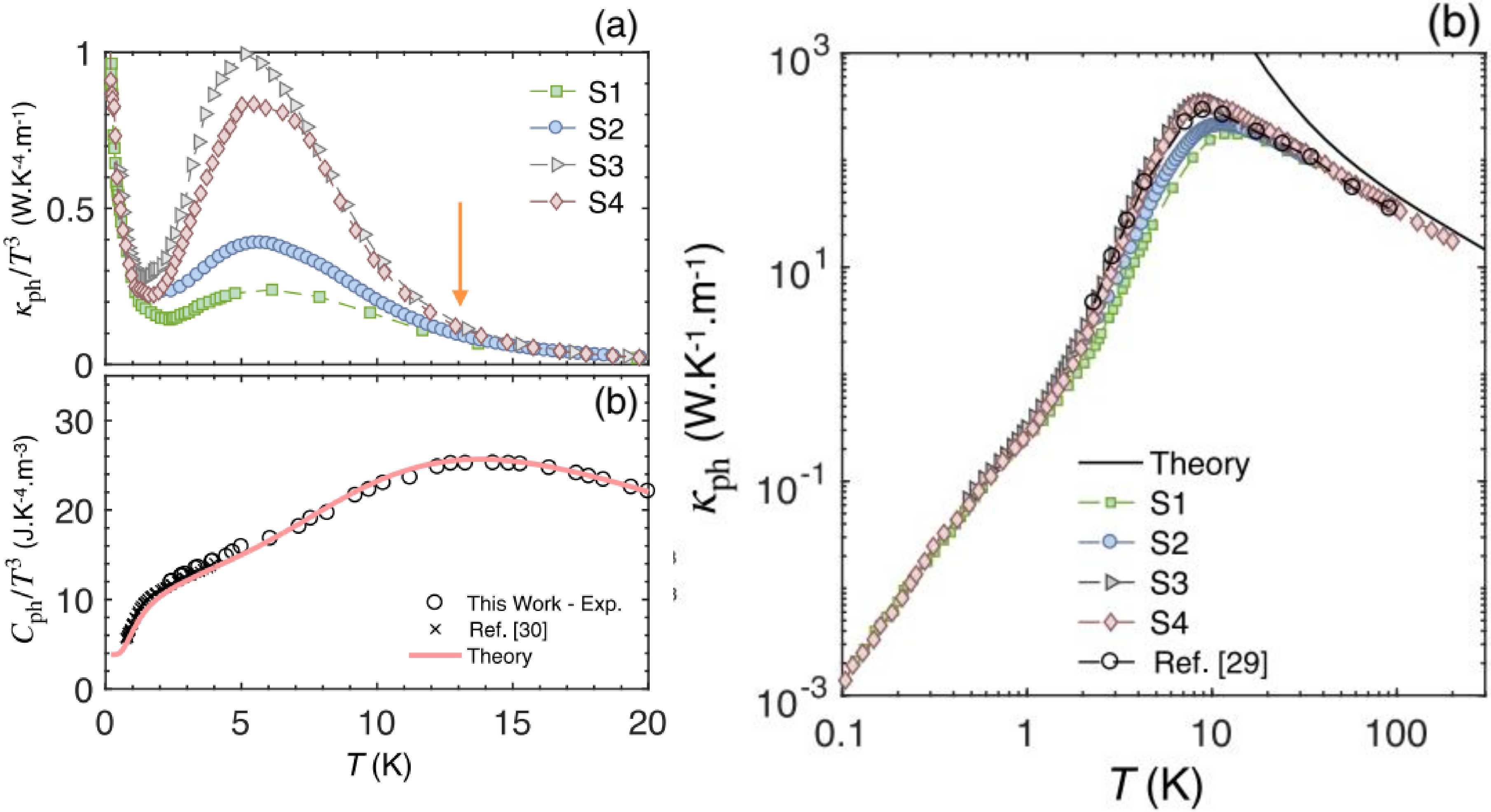


$$\rho = \rho_0 + T^{\gamma}$$

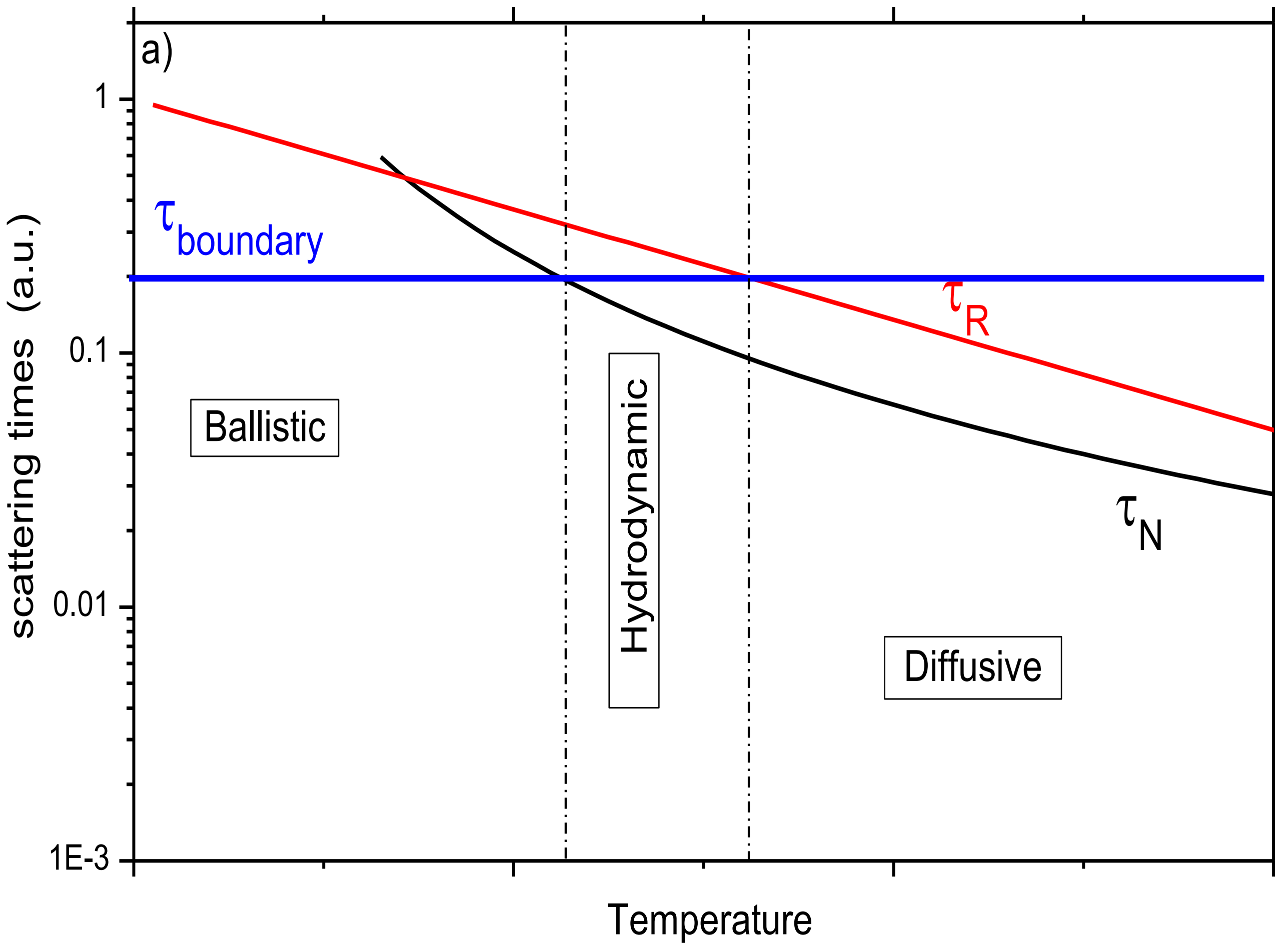


- In Sb, γ never attains 5.
- In Mo and W, γ shoots up to 5, but becomes 2 at the end.
- The prefactor of the T-square resistivity scales with the Fermi temperature.

Specific heat and thermal conductivity of phonons in Sb

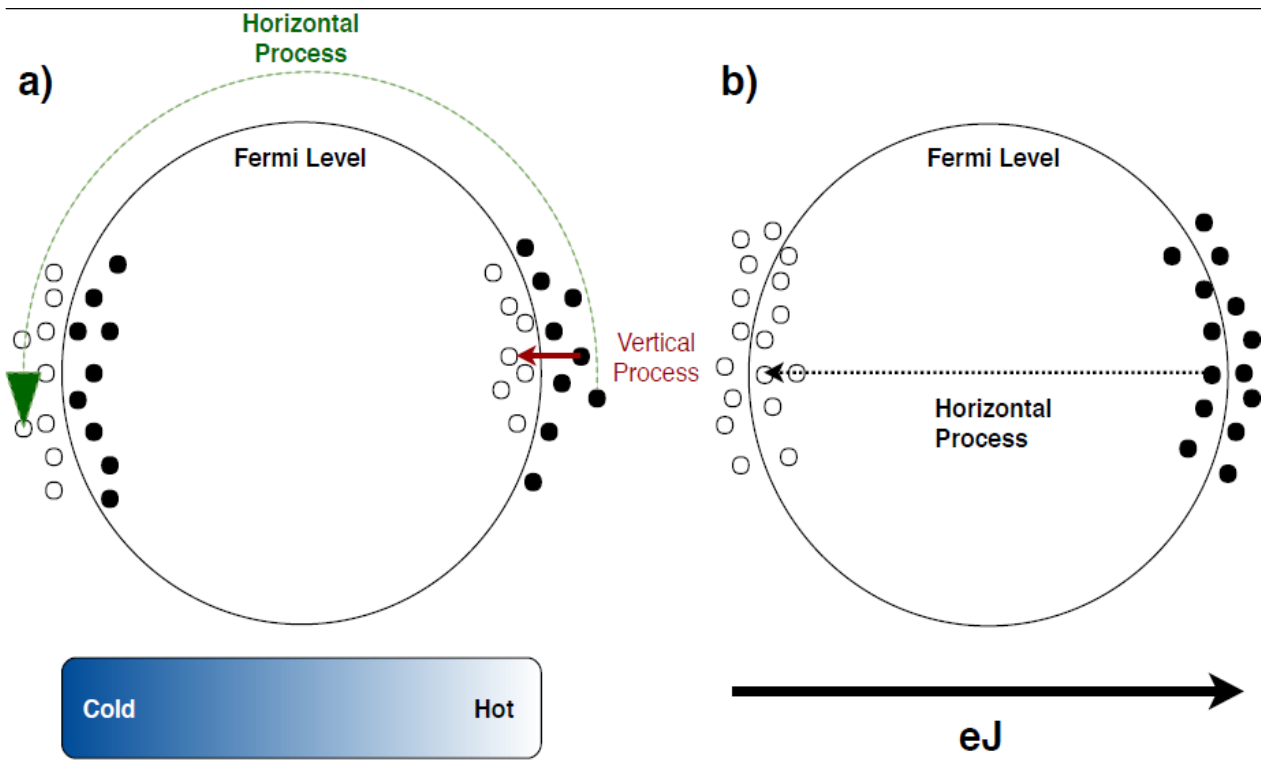


The hydrodynamic window requires a specific hierarchy!



- **Abundant normal** scattering
- **Intermediate boundary** scattering
- **Small resistive** scattering

Inelastic scattering induces a deviation

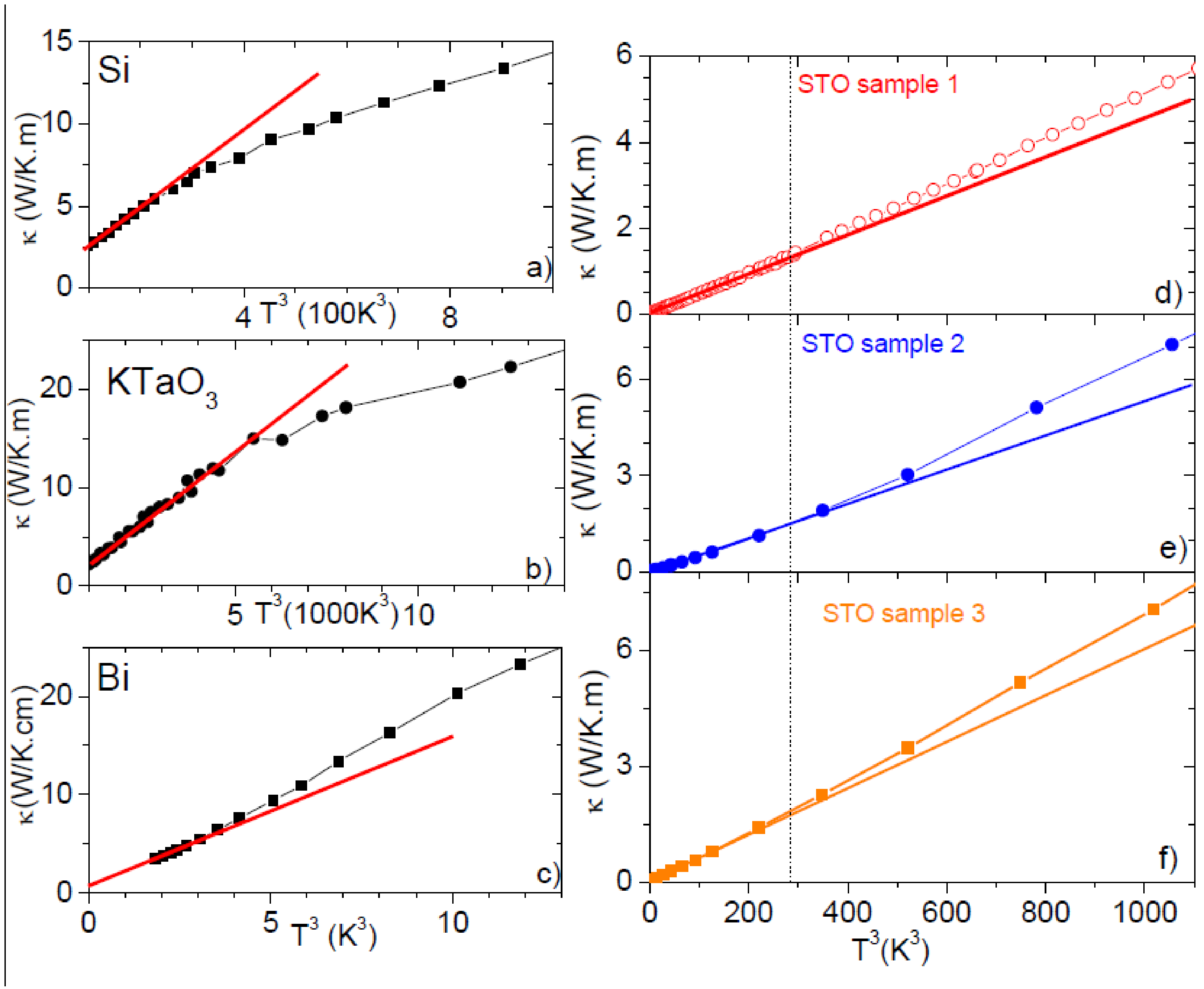


Small-angle scattering less efficiently the charge current!

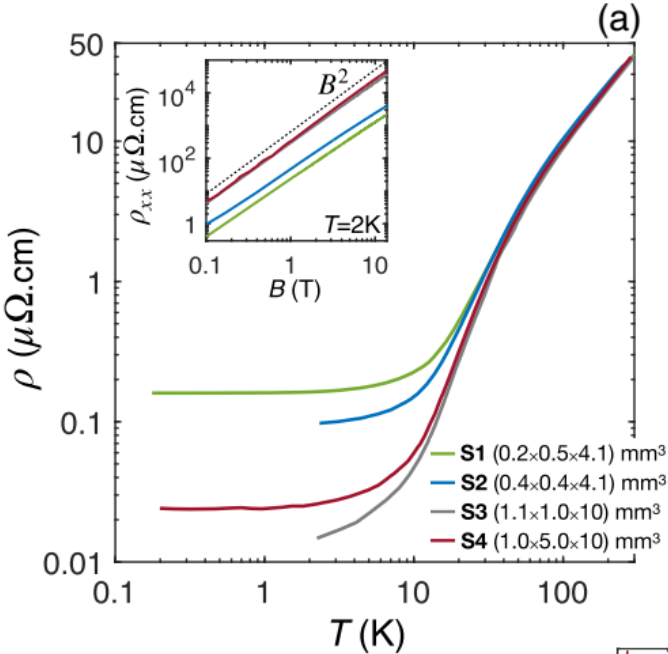
A pondering factor for momentum current but absent for energy current:

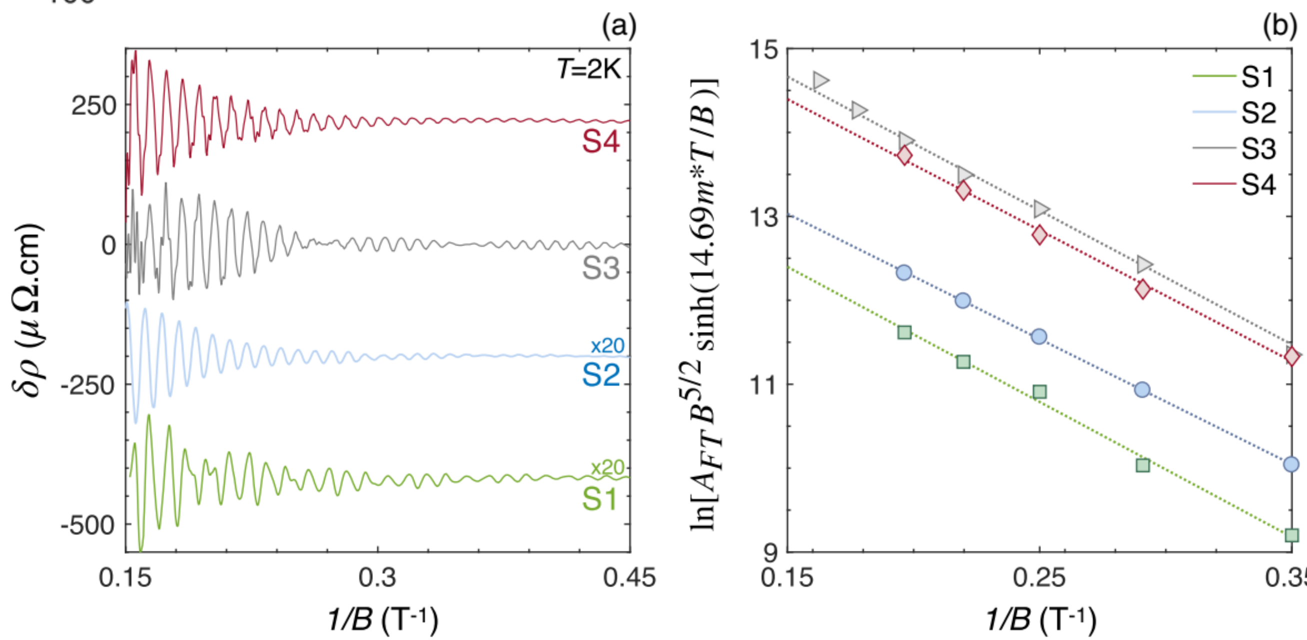
$1 - \cos\Theta$

Faster than T^3 thermal conductivity



Dingle mobility is much smaller than transport mobility in Sb





Prediction of hydrodynamics in perfectly compensated metals

SOVIET PHYSICS – SOLID STATE VOL. 8, NO. 10 APRIL, 1967

THEORY OF THE SECOND SOUND IN SEMICONDUCTORS

L. É. Gurevich and B. I. Shklovskii

A. F. Ioffe Physicotechnical Institute, Academy of Sciences of the USSR, Leningrad

Translated from Fizika Tverdogo Tela, Vol. 8, No. 10,
pp. 3050–3055, October, 1966
Original article submitted February 25, 1966;
revision submitted April 25, 1966

Charge carriers provide a momentum reservoir for normal collisions between phonons!

SOVIET PHYSICS JETP VOLUME 28, NUMBER 3 MARCH, 1969

ELECTRON SOUND IN METALS

R. N. GURZHI and V. M. KONTOROVICH

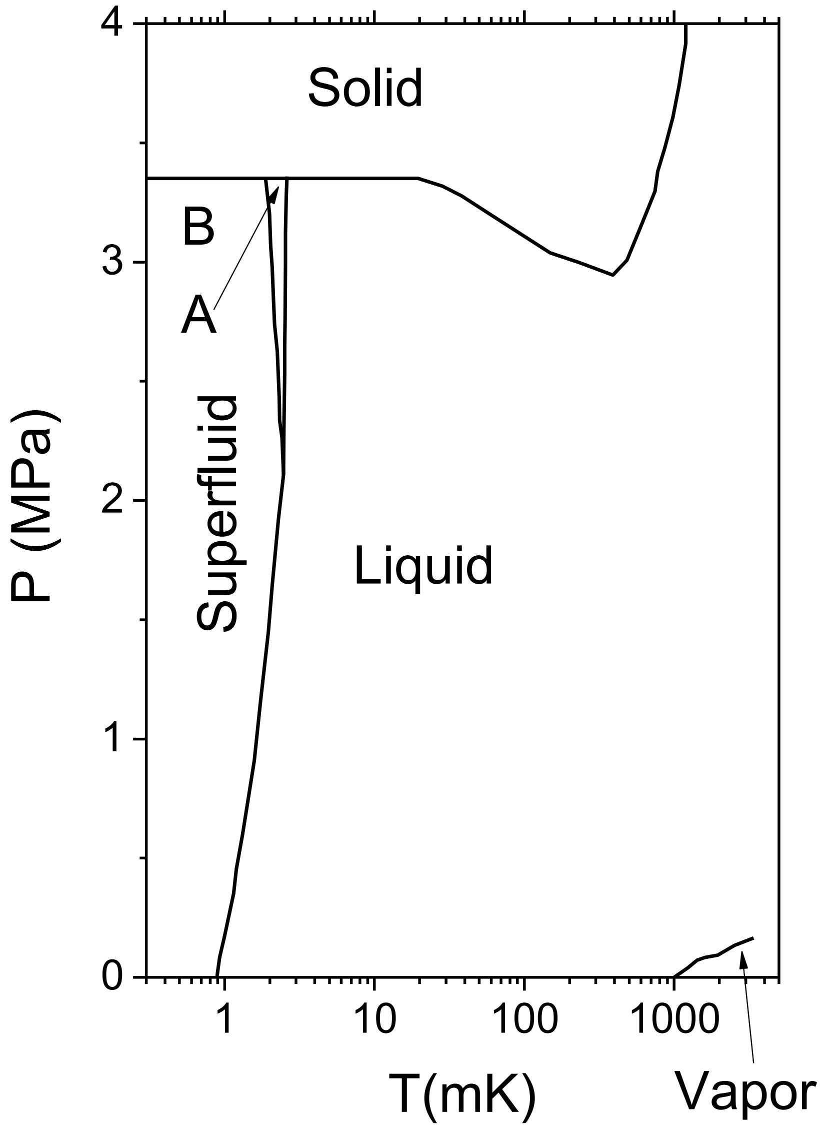
Physico-technical Institute, Ukrainian Academy of Sciences; Institute of Radiophysics and Electronics, Ukrainian Academy of Sciences

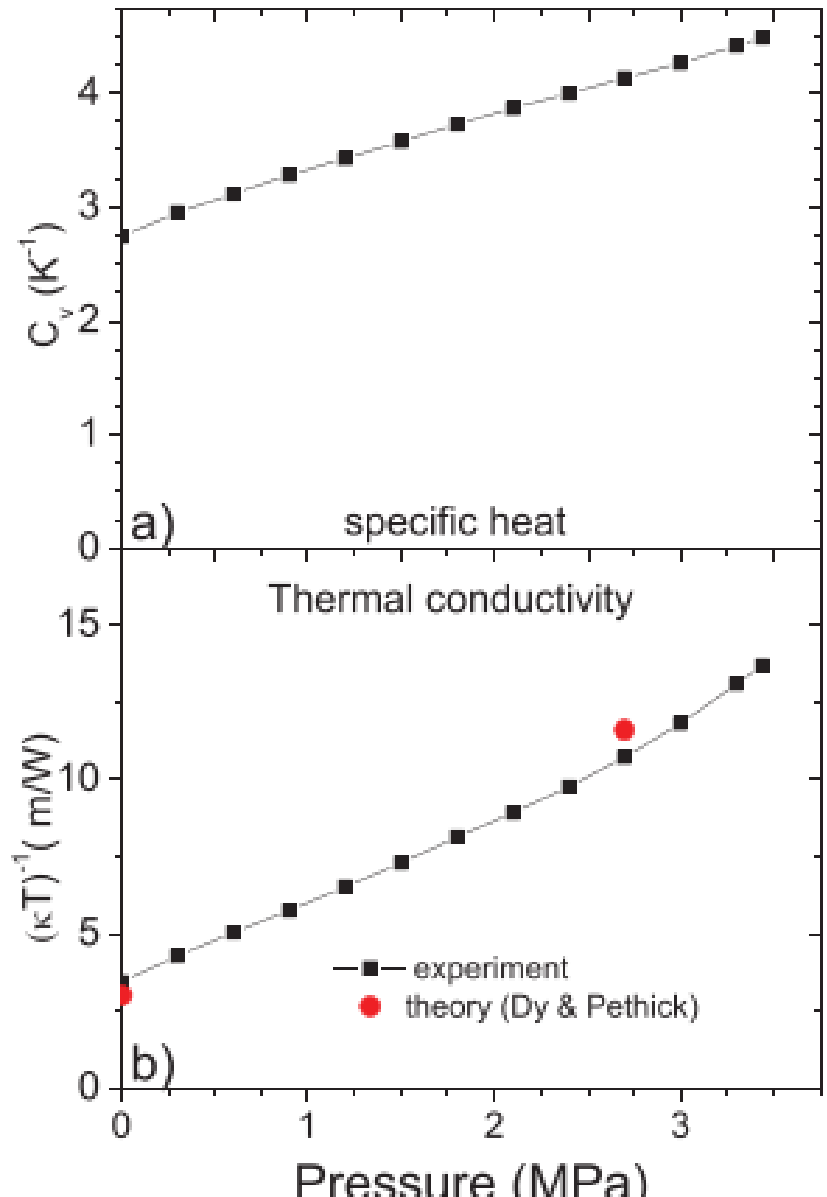
Submitted April 18, 1968

Zh. Eksp. Teor. Fiz. 55, 1105–1116 (September, 1968)

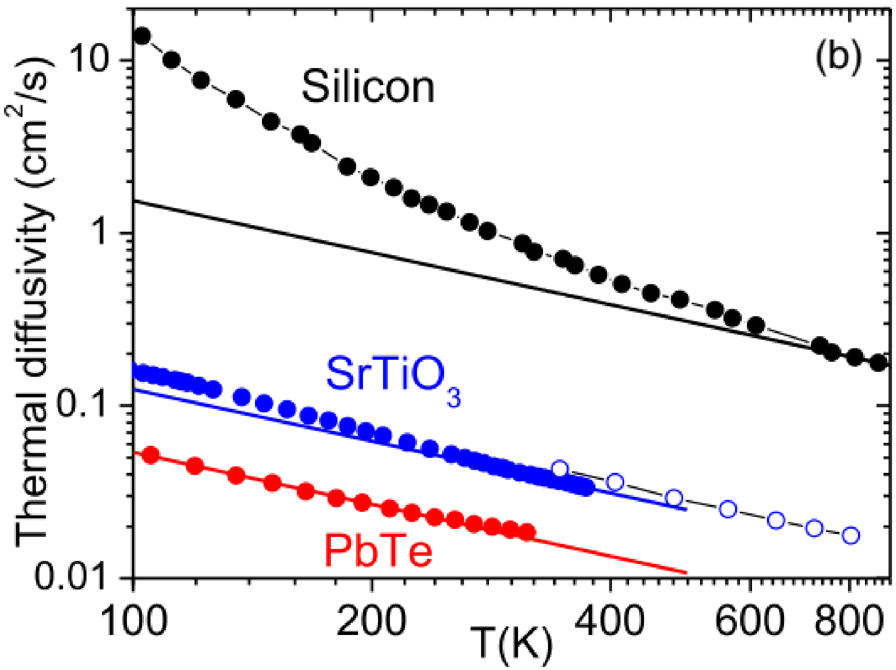
Phonons provide a momentum reservoir for normal collisions between electrons and holes!

^{3}He under pressure





Planckian disspiation in the kinetic regime



System	$D_{300\ K}$ (mm^2/s)	v_{sl} (100)(km/s)	s
$SrTiO_3$	4.0	7.87	2.6
PbTe	1.9	3.59	5.9
Si	91	8.43	51

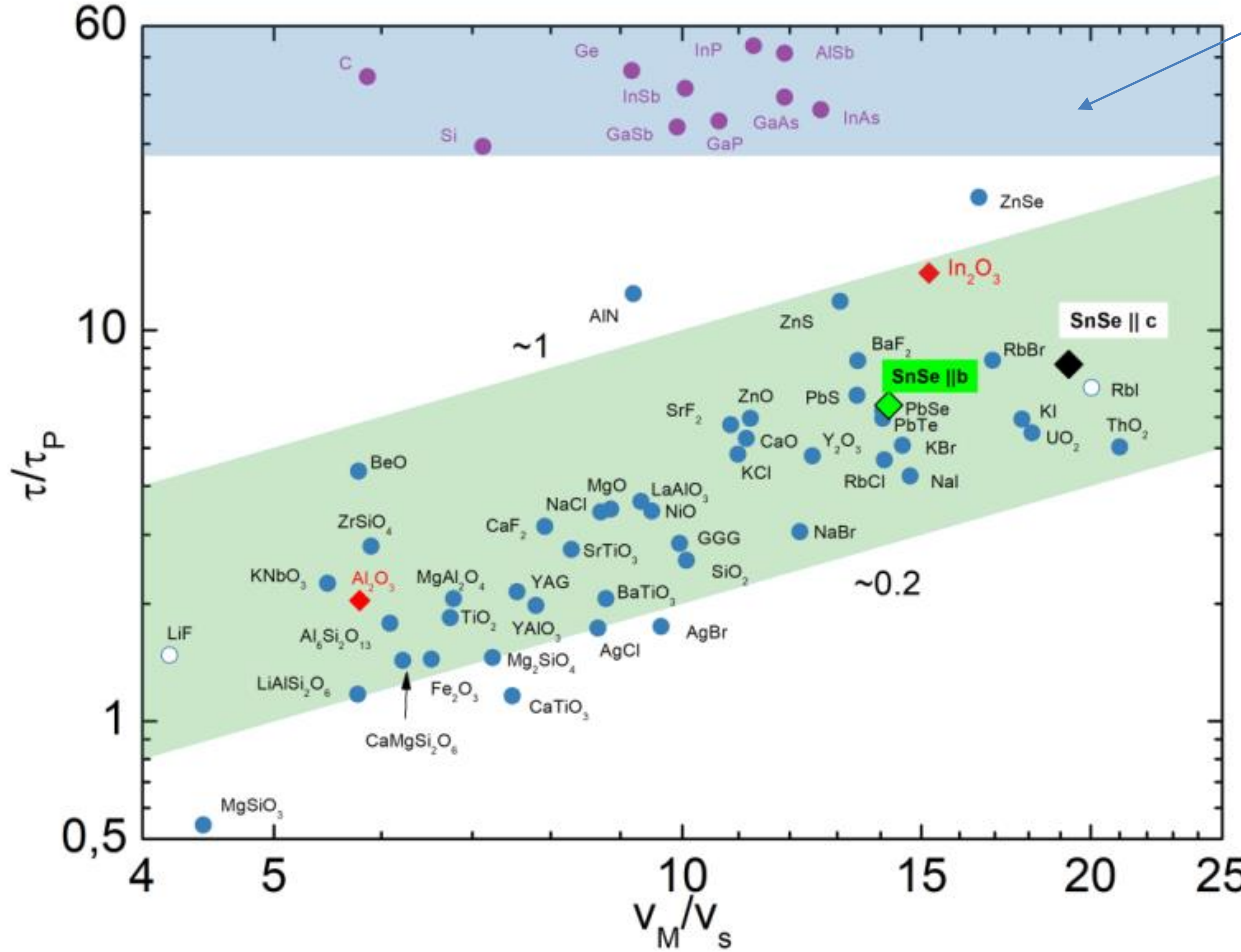
$$D = s v_s^2 \tau_p. \qquad \tau_p = (\hbar / k_B T)$$

- Thermal resisistivity of insulators is linear in temperature!
- The scattering time approaches the Planckian time!

- Zhang et al. PNAS 114, 5378 (2017).
- Behnia & Kapitulnik, J. Phys.: Condens. Matter 31, 405702 (2019).
- Mousatov & Hartnoll, Nat. Phys. 16, 579 (2020)

Mousatov-Hartnoll plot

Why these solids do not fit in?



Quantum chaos? See Berg and Tulipman, PHYSICAL REVIEW RESEARCH 2, 033431 (2020)

Scattering time to planckian time ratio scales with melting velocity to the sound velocity ratio!

Chaotic phonons?

PHYSICAL REVIEW B VOLUME 61, NUMBER 15 15 APRIL 2000-I

Correlations in optical phonon spectra of complex solids

G. Fagas, Vladimir I. Fal'ko, and C. J. Lambert
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Yuval Gefen
Department of Condensed Matter Physics, Weizmann Institute of Science, 76100 Rehovot, Israel
(Received 14 December 1999)

Spectral correlations in the optical phonon spectrum of a solid with a complex unit cell are analyzed using the Wigner-Dyson statistical approach. Despite the fact that all force constants are real, we find that the statistics are predominantly of the GUE type depending on the location within the Brillouin zone of a crystal and the unit cell symmetry. Analytic and numerical results for the crossover from GOE to GUE statistics are presented.

- Melting is non-linear
- Chaos emerges above a threshold number of degrees of freedom
- Contrast a simple pendulum with a double pendulum