

Two-point correlation function for Dirichlet L -functions

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2013 J. Phys. A: Math. Theor. 46 095202

(<http://iopscience.iop.org/1751-8121/46/9/095202>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.175.97.14

The article was downloaded on 11/07/2013 at 15:11

Please note that [terms and conditions apply](#).

Two-point correlation function for Dirichlet L -functions

E Bogomolny^{1,2} and J P Keating³

¹ Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris-Sud, Orsay, F-91405, France

² CNRS, UMR8626, Orsay, F-91405, France

³ School of Mathematics, University of Bristol, Bristol BS8 1TW, UK

E-mail: j.p.keating@bristol.ac.uk

Received 1 November 2012, in final form 21 January 2013

Published 12 February 2013

Online at stacks.iop.org/JPhysA/46/095202

Abstract

The two-point correlation function for the zeros of Dirichlet L -functions at a height E on the critical line is calculated heuristically using a generalization of the Hardy–Littlewood conjecture for pairs of primes in arithmetic progression. The result matches the conjectured random-matrix form in the limit as $E \rightarrow \infty$ and, importantly, includes finite- E corrections. These finite- E corrections differ from those in the case of the Riemann zeta-function, obtained in Bogomolny and Keating (1996 *Phys. Rev. Lett.* **77** 1472), by certain finite products of primes which divide the modulus of the primitive character used to construct the L -function in question.

PACS numbers: 02.10.De, 03.65.Sq, 02.10.Yn

1. Introduction

The observation that statistical properties of the zeros of the Riemann zeta function are the same as those of the eigenvalues of unitary ensembles of random matrices has a long history [1, 2] (see [3] for a review). The main difficulty in proving this is the calculation of off-diagonal prime contributions to correlation functions when these are represented using the explicit formula that relates the zeros to the primes. It appears that to calculate these contributions it is necessary to use the Hardy–Littlewood conjecture [4] for pairs of prime numbers. In [5] a smooth version of this conjecture was used to demonstrate that the two-point correlation function of the Riemann zeros in the universal limit coincides with that of the Unitary Ensembles (e.g. the Gaussian Unitary Ensembles (GUE), or Circular Unitary Ensembles (CUE)) of random matrix theory (RMT). In [6], using the same smoothed form of the Hardy–Littlewood conjecture, it was shown that all correlation functions of the Riemann zeros agree with the corresponding GUE/CUE results.

A more precise expression for the two-point correlation function of the Riemann zeta function zeros was obtained in [7] using another method which is equivalent to the full

Hardy–Littlewood conjecture (details of the calculations can be found in [8]). The conjectured expression for the two-point correlation function reads

$$R_2(\epsilon) = \bar{d}^2(E) + R_2^{(\text{diag})}(\epsilon) + R_2^{(\text{off})}(\epsilon). \tag{1}$$

Here, if $d(E)$ denotes the density of zeros at height E on the critical line and

$$\bar{d}(E) = \frac{1}{2\pi} \ln \left(\frac{E}{2\pi} \right) \tag{2}$$

its asymptotic mean (defined to be the derivative of the non-fluctuating part of the counting function of the zeros, $N(E) = \frac{E}{2\pi} \ln \left(\frac{E}{2\pi} \right) - \frac{E}{2\pi} + O(\ln E)$), then

$$R_2(\epsilon) = \langle d(E - \epsilon/2) d(E + \epsilon/2) \rangle. \tag{3}$$

The diagonal part of the correlation function, $R_2^{(\text{diag})}(\epsilon)$, has the form

$$R_2^{(\text{diag})}(\epsilon) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon^2} \ln[|\zeta(1 + i\epsilon)|^2 \Phi^{(\text{diag})}(\epsilon)], \tag{4}$$

where

$$\Phi^{(\text{diag})}(\epsilon) = \prod_p \exp \left(2 \sum_{m=1}^{\infty} \frac{1-m}{m^2 p^m} \cos(m\epsilon \ln p) \right). \tag{5}$$

The off-diagonal part of the correlation function, $R_2^{(\text{off})}(\epsilon)$ is

$$R_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} e^{2\pi i \epsilon \bar{d}(E)} |\zeta(1 + i\epsilon)|^2 \Phi^{(\text{off})}(\epsilon) + \text{c.c.}, \tag{6}$$

where

$$\Phi^{(\text{off})}(\epsilon) = \prod_p \left(1 - \frac{(p^{i\epsilon} - 1)^2}{(p - 1)^2} \right). \tag{7}$$

The assumed (optimistic) precision of these formulas is $O(E^{-1/2})$ and they agree very well with numerical calculations of Odlyzko [9, 10]. In the limit $E \rightarrow \infty$, $R_2(\epsilon/\bar{d}(E))/(\bar{d}(E))^2$ tends to the GUE/CUE expression for the two-point correlation function.

In [11] the method of [5] was generalized to show that the two-point correlation function of the zeros of a given Dirichlet L -function tends, in the universal limit (i.e. infinitely high up the critical line), to the GUE/CUE result.

The purpose of this note is to obtain an expression, similar to the above formulas, for the two-point correlation function of Dirichlet L -functions using the full Hardy–Littlewood conjecture. The final answer is similar to the Riemann case (4)–(7) but with a finite contribution from primes which divide the modulus of the character defining the L -function in question. This demonstrates that the method of [7] applies in much greater generality than to the zeta function alone: it extends to other L -functions. The plan of the note is as follows. In section 2 the appropriate form of the conjecture for prime pairs in arithmetical progressions is obtained. In section 3 the explicit formula for Dirichlet L -functions is stated and in section 4 the formal expression for the two-point correlation function of their zeros is presented. In section 4.1 the diagonal contribution is derived and in section 4.2 the prime pairs conjecture is used to derive the off-diagonal contribution. The list of final formulas is given in section 5. In the appendix it is checked that the GUE/CUE result corresponds to the contribution of only one term, as shown in [11] by a different method.

For the Riemann zeta-function, the expression for the two-point correlation function, (4)–(7), that includes the lower order terms may also be obtained by a heuristic recipe [12, 13] for calculating averages of products and ratios of zeta-functions [14]. That approach extends to L -functions too. It has the advantage of being simple and powerful in its generality, but

the disadvantage that some of the key steps can only be justified under the assumption that certain large error terms must cancel each other, for reasons not yet understood. Our goal in this work is to extend the explicit approach based on the Hardy–Littlewood conjecture to the more general case of L -functions.

2. Prime pairs in arithmetical progression

Let $\Lambda(m)$ denote the von Mangoldt function: $\Lambda(m) = \log p$ if m is a power of a prime p and $\Lambda(m) = 0$ otherwise. The usual Hardy–Littlewood conjecture for the density of prime pairs can be written in two equivalent forms [4]: either

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N \Lambda(m) \Lambda(m+h) = \alpha(h), \tag{8}$$

or, if $\pi_2(h; N)$ denotes the number of primes $p < N$ such that $p+h$ is also a prime, then

$$\pi_2(h; N) \xrightarrow{N \rightarrow \infty} \alpha(h) \frac{N}{\ln^2 N}. \tag{9}$$

Here $\alpha(h)$ is defined as follows: $\alpha(h) = 0$ if h is odd, and if h is even

$$\alpha(h) = \sum_{q=1}^{\infty} \left(\frac{\mu(q)}{\phi(q)} \right)^2 c_q(h) = 2C_2 \prod_{p|h} \frac{p-1}{p-2}, \tag{10}$$

where $c_q(n)$ are the Ramanujan coefficients

$$c_q(n) = \sum_{(p,q)=1} e^{2\pi i pn/q}, \tag{11}$$

$\mu(q)$ is the Möbius function, $\phi(q)$ is Euler’s totient function, and C_2 is the twin prime constant

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) \approx 0.6601618. \tag{12}$$

To calculate the off-diagonal part of the two-point correlation function of the zeros of a Dirichlet L -function we need to know the mean value of the product of two Λ -functions in arithmetic progression

$$\alpha(r_2 - r_1, k) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N \Lambda(km + r_1) \Lambda(km + r_2). \tag{13}$$

It is plain that in order that progressions $mk + r_i$ contain an infinite number of primes it is necessary that $(k, r_1) = 1$ and $(k, r_2) = 1$. In the above definition we anticipate that under these conditions the indicated limit depends only on the difference between r_2 and r_1 .

This limit can heuristically be obtained from usual probabilistic considerations [5], but we found it more convenient to start with the exact expansion of the Λ -function as a series of the Ramanujan coefficients [15]

$$\frac{\phi(n)\Lambda(n)}{n} = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n). \tag{14}$$

Using the following properties of the Ramanujan coefficients

$$\text{if } m \equiv n \pmod{q}, \quad c_q(m) = c_q(n), \tag{15}$$

$$\text{if } (q, r) = 1, \quad c_{qr}(n) = c_q(n)c_r(n), \tag{16}$$

one can easily prove that for squarefree integers q_1 and q_2 (as we will multiply them by $(\mu(q_1)\mu(q_2))^2$) one has

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N c_{q_1}(km + r_1)c_{q_2}(km + r_2) = c_v(r_1 - r_2)c_{\tilde{q}_1}(r_1)c_{\tilde{q}_2}(r_2), \quad (17)$$

where $q_i = v\tilde{q}_i$ and $(v, k) = 1$ but $\tilde{q}_1 \nmid k$ and $\tilde{q}_2 \nmid k$. In all other cases the above mean value is zero.

Using (14) and formally interchanging the summation and the average (which is not justified rigorously) after simple calculations we get

$$\alpha(r_2 - r_1, k) = S(k) \sum_{(q,k)=1} \left(\frac{\mu(q)}{\phi(q)} \right)^2 c_q(r_2 - r_1) \quad (18)$$

with

$$\begin{aligned} S(k) &= \sum_{v_1 \mid k} \frac{\mu(v_1)}{\phi(v_1)} c_{v_1}(r_1) \sum_{v_2 \mid k} \frac{\mu(v_2)}{\phi(v_2)} c_{v_2}(r_2) \\ &= \prod_{p \mid k} \left(1 - \frac{c_p(r_1)}{\phi(p)} \right) \prod_{p \mid k} \left(1 - \frac{c_p(r_2)}{\phi(p)} \right) \\ &= \begin{cases} \prod_{p \mid k} \left(\frac{p}{p-1} \right)^2 = \left(\frac{k}{\phi(k)} \right)^2, & (r_1, k) = 1, (r_2, k) = 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (19)$$

In the last step we used that for prime p

$$c_p(n) = \begin{cases} -1, & \text{if } p \nmid n \\ \phi(p), & \text{if } p \mid n. \end{cases} \quad (20)$$

The expression (19) implies that the Hardy–Littlewood conjecture for prime pairs in arithmetical progression can be formulated as follows. Consider an arithmetical progression $km + r$ with $(k, r) = 1$, the number of primes in this progression with $m < N$ such that $p + h$ is also a prime ($(k, r + h) = 1$) is asymptotically

$$\left(\# p, p + h \text{ primes, } p \in mk + r, m < N \right) \xrightarrow{N \rightarrow \infty} \alpha(h, k) \frac{N}{\ln^2 N}. \quad (21)$$

The expression for $\alpha(h, k)$ as a product of prime numbers has the form

$$\alpha(h, k) = \alpha(h)\beta(h, k), \quad (22)$$

where $\alpha(h)$ is the same as in the usual Hardy–Littlewood conjecture (10) and $\beta(h, k)$ is a finite product of primes which divide k ,

$$\beta(h, k) = \prod_{\substack{p \mid k \\ p \nmid h}} \frac{p-1}{p-2} \prod_{p \mid k} \frac{(p-1)^2}{p(p-2)}. \quad (23)$$

3. Explicit formula

Dirichlet L -functions are defined as follows

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s} \right)^{-1}, \quad (24)$$

where $\chi(n)$ is a character modulo an integer k . Below we shall assume that the character is primitive. The generalized Riemann hypothesis (GRH) states that all non-trivial zeros of this L -function lie on the critical line $s = 1/2 + iE$ with real E ; that is, the zeros may be written $s = 1/2 + iE_n$ where E_n is real. We are interested in the statistical properties of these zeros as $E \rightarrow \infty$.

The explicit formula for the density of zeros of any Dirichlet L -function has the form

$$d(E) = \sum_n \delta(E - E_n) = \bar{d}(E) + d^{(osc)}(E), \tag{25}$$

where the smooth part is, asymptotically,

$$\bar{d}(E) = \frac{1}{2\pi} \ln\left(\frac{kE}{2\pi}\right) \tag{26}$$

and the oscillating part is

$$d^{(osc)}(E) = -\frac{1}{2\pi} \sum_n \frac{\Lambda(n)}{\sqrt{n}} \chi(n) e^{iE \ln n} + \text{c.c.} \tag{27}$$

4. Two-point correlation function

The formal expression for the two-point correlation function is

$$R_2(\epsilon_1, \epsilon_2) = \langle d(E + \epsilon_1) d(E + \epsilon_2) \rangle \approx \bar{d}^2(E) + R_2^{(osc)}(\epsilon_1, \epsilon_2) \tag{28}$$

with

$$\begin{aligned} R_2^{(osc)}(\epsilon_1, \epsilon_2) &= \langle d^{(osc)}(E + \epsilon_1) d^{(osc)}(E + \epsilon_2) \rangle \\ &= \frac{1}{4\pi^2} \left\langle \sum_{n_1, n_2} \frac{\Lambda(n_1)\Lambda(n_2)}{\sqrt{n_1 n_2}} \chi(n_1) \bar{\chi}(n_2) e^{i(E+\epsilon_1) \ln n_1 - i(E+\epsilon_2) \ln n_2} \right\rangle + \text{c.c.} \\ &= R_2^{(diag)}(\epsilon_1, \epsilon_2) + R_2^{(off)}(\epsilon_1, \epsilon_2). \end{aligned} \tag{29}$$

4.1. Diagonal contribution

The diagonal part of the two-point function consists of taking into account only the terms with $n_1 = n_2 = n$

$$R_2^{(diag)}(\epsilon_1, \epsilon_2) = \frac{1}{4\pi^2} \sum_n \frac{\Lambda^2(n)}{n} |\chi(n)|^2 e^{i(\epsilon_1 - \epsilon_2) \ln n} + \text{c.c.} \tag{30}$$

For any character $|\chi(n)|^2 = 1$ for $(n, k) = 1$ and $|\chi(n)|^2 = 0$ for $(n, k) > 1$. Therefore, setting $\epsilon = \epsilon_1 - \epsilon_2$,

$$R_2^{(diag)}(\epsilon) = \frac{1}{4\pi^2} \sum_{(n,k)=1} \frac{\Lambda^2(n)}{n} e^{i\epsilon \ln n} + \text{c.c.} \tag{31}$$

The calculation of this expression can be carried out exactly as for the Riemann case, but removing terms that divide k :

$$R_2^{(diag)}(\epsilon) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon^2} \ln[|\zeta(1 + i\epsilon)|^2 \Phi^{(diag)}(\epsilon) \Psi(\epsilon, k)], \tag{32}$$

where $\Phi^{(diag)}(\epsilon)$ is the same function as in the Riemann case (5) and

$$\Psi(\epsilon, k) = \exp\left(-\sum_{p|k} \sum_{m=1}^{\infty} \frac{1}{m^2 p^m} e^{im\epsilon \ln p} + \text{c.c.}\right). \tag{33}$$

4.2. Off-diagonal contribution

The off-diagonal part of the correlation function corresponds to $n_1 = n$, $n_2 = n + h$ with $h \ll n$,

$$R_2^{(\text{off})}(\epsilon) \approx \frac{1}{4\pi^2} \left\langle \sum_{n,h} \frac{\Lambda(n)\Lambda(n+h)}{n} \chi(n)\bar{\chi}(n+h) e^{i\epsilon \ln n - iEh/n} \right\rangle + \text{c.c.} \quad (34)$$

Now choose $n = km + r_1$ and $h = kl + r$ with r_1 and r modulo k (but, of course, $r_1 \neq 0$ and $r + r_1 \neq 0 \pmod k$)

$$R_2^{(\text{off})}(\epsilon) \approx \frac{1}{4\pi^2} \sum_{r_1, r=0}^{k-1} \chi(r_1)\bar{\chi}(r_1+r) \times \sum_{m,l} \frac{\Lambda(km+r_1)\Lambda(km+r_1+kl+r)}{km} e^{i\epsilon \ln km - iE(kl+r)/(km)} + \text{c.c.} \quad (35)$$

Using (18) for the mean value of the product of the two Λ -functions one gets

$$R_2^{(\text{off})}(\epsilon) \approx \frac{1}{4\pi^2} S(k) \sum_{r_1, r=0}^{k-1} \chi(r_1)\bar{\chi}(r_1+r) \times \sum_{(q,k)=1} \sum_{(p,q)=1} \left(\frac{\mu(q)}{\phi(q)} \right)^2 \sum_{m,l} \frac{1}{km} e^{i\epsilon \ln km - iE(kl+r)/(km) + 2\pi i p(kl+r)/q} + \text{c.c.} \quad (36)$$

As in the Riemann case, the summation over l and m can be performed as follows. The sum over all l gives, using the Poisson Summation formula,

$$\sum_{l=-\infty}^{\infty} e^{2\pi i(kl+r)x} = \sum_{s=-\infty}^{\infty} \delta(kx - s) e^{2\pi i r x} = \sum_{s=-\infty}^{\infty} \delta(kx - s) e^{2\pi i s r/k} \quad (37)$$

with

$$x = \frac{p}{q} - \frac{E}{2\pi km}. \quad (38)$$

In the above formulas p is coprime to q and smaller than q , i.e. $0 < p < q$. To take into account the periodicity of the δ -function above it is convenient to define the integer

$$R = kp - qs. \quad (39)$$

It is coprime to q because p and k are coprime to q .

By construction $(k, q) = 1$. Therefore there exist two integers a and b such that $(a, q) = 1$, $(b, k) = 1$ and

$$ak - bq = 1. \quad (40)$$

Any number M can be written as $M = pk - qs$. Integers p and s are not unique but have the form $p = aM + tq$ and $s = bM + tk$. If we require that $(M, q) = 1$ then uniquely $aM = r_m + qt_m$ with $(r_m, q) = 1$ and $0 < r_m < q$. Now we get uniquely $p = r_m$ and $t = -t_m$. This means that any integer R coprime with q can be uniquely rewritten in the form (39) with $0 < p < q$, and the summation over $(p, q) = 1$ with $0 < p < q$ and all s is equivalent to the summation over $(R, q) = 1$ without restriction on R .

One has

$$\sum_{m,l} \frac{1}{m} e^{i\epsilon \ln km - iE(kl+r)/m + 2\pi i p(kl+r)/q} = \int \frac{dm}{m} e^{i\epsilon \ln km} \sum_s \delta\left(\frac{E}{2\pi m} - k\frac{p}{q} - s\right) e^{2\pi i sr/k}$$

$$\approx e^{i\epsilon \ln(Ek/2\pi)} \sum_s \left(\frac{q}{kp - qs}\right)^{1+i\epsilon} e^{2\pi i sr/k}. \tag{41}$$

It leads to the following expression for the two-point function:

$$R_2^{(\text{off})}(\epsilon) = \frac{S(k)}{4\pi^2 k} e^{i\epsilon \ln(Ek/2\pi)} \sum_{r=0}^{k-1} g(r) \sum_{(q,k)=1} \sum_{\substack{s \\ (p,q)=1}} \left(\frac{q}{kp - qs}\right)^{1+i\epsilon} e^{2\pi i sr/k} \left(\frac{\mu(q)}{\phi(q)}\right)^2 + \text{c.c.}, \tag{42}$$

where

$$g(r) = \sum_{r_1=0}^{k-1} \chi(r_1) \bar{\chi}(r_1 + r). \tag{43}$$

Here we will need the condition that the character χ is assumed to be primitive. Consider the Gauss sum for a given character χ

$$\tau(\chi) = \sum_{m=1}^k \chi(m) e^{2\pi i m/k}. \tag{44}$$

It is known that for primitive characters the following is true for all s (see e.g. [16] section 9)

$$\bar{\chi}(s) \tau(\chi) = \sum_{m=1}^k \chi(m) e^{2\pi i ms/k} \tag{45}$$

and

$$|\tau(\chi)|^2 = k. \tag{46}$$

Using these formulas we get

$$\sum_{r=0}^{k-1} g(r) e^{2\pi i rs/k} = \sum_{r_1, r_2=0}^{k-1} \chi(r_1) \bar{\chi}(r_2) e^{2\pi i (r_2 - r_1)s/k} = |\chi(s)|^2 |\tau(\chi)|^2 = \begin{cases} k, & (k, s) = 1 \\ 0, & (k, s) > 1. \end{cases} \tag{47}$$

Therefore, s has to be coprime to k , $(s, k) = 1$ and the integer $R = kp - ks$ will be coprime to both q and k . As $(q, k) = 1$ these restrictions can be written in the form: $(R, qk) = 1$.

Finally, the two-point function for the Dirichlet L -function with a primitive character is

$$R_2^{(\text{off})}(\epsilon) = \frac{S(k)}{4\pi^2} e^{i\epsilon \ln(Ek/2\pi)} \sum_{(q,k)=1} \sum_{(R,qk)=1} \left(\frac{q}{R}\right)^{1+i\epsilon} \left(\frac{\mu(q)}{\phi(q)}\right)^2 + \text{c.c.} \tag{48}$$

The summation over all integers R coprime with qk is performed by the inclusion–exclusion principle

$$\sum_{(R,qk)=1} f(R) = \sum_{t=1}^{\infty} \sum_{\delta|qk} f(t\delta) \mu(\delta). \tag{49}$$

This gives

$$\sum_{(R,qk)=1} \frac{1}{R^{1+i\epsilon}} = \zeta(1+i\epsilon) \prod_{p|q} \left(1 - \frac{1}{p^{1+i\epsilon}}\right) \prod_{p|k} \left(1 - \frac{1}{p^{1+i\epsilon}}\right). \tag{50}$$

The remaining sum over q coprime to k is calculated as follows:

$$\begin{aligned} \sum_{(q,k)=1} q^{1+i\epsilon} \left(\frac{\mu(q)}{\phi(q)}\right)^2 \prod_{p|q} \left(1 - \frac{1}{p^{1+i\epsilon}}\right) &= \prod_{(p,k)=1} \left(1 + \frac{1}{p^{1-i\epsilon}} \frac{1}{(1-1/p)^2} \left(1 - \frac{1}{p^{1+i\epsilon}}\right)\right) \\ &= \prod_{(p,k)=1} \frac{1}{1 - p^{-1+i\epsilon}} \left(1 - \frac{(p^{i\epsilon} - 1)^2}{(p-1)^2}\right) \\ &= \zeta(1-i\epsilon) \Phi^{(\text{off})}(\epsilon) \prod_{p|k} \left(\frac{1 - p^{-1+i\epsilon}}{1 - \frac{(p^{i\epsilon} - 1)^2}{(p-1)^2}}\right) \\ &= \zeta(1-i\epsilon) \Phi^{(\text{off})}(\epsilon) \prod_{p|k} \frac{p-1}{p(1 + (p^{i\epsilon} - 1)/(p-1))}. \end{aligned} \tag{51}$$

Here $\Phi^{(\text{off})}(\epsilon)$ is the same function as in the Riemann case (7).

Taking into account the expression (19) for $S(k)$, we find that the total off-diagonal contribution to the two-point correlation function is

$$R_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} e^{i\epsilon \ln(Ek/2\pi)} |\zeta(1+i\epsilon)|^2 \Phi^{(\text{off})}(\epsilon) \Psi^{(\text{off})}(\epsilon, k) + \text{c.c.}, \tag{52}$$

where

$$\Psi^{(\text{off})}(\epsilon, k) = \prod_{p|k} \left(1 + \frac{(p^{i\epsilon/2} - p^{-i\epsilon/2})^2}{p - p^{-i\epsilon}}\right)^{-1}. \tag{53}$$

Notice that $\Psi^{(\text{off})}(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 1$ so the universal limit of the two-point correlation function coincides with the GUE/CUE result.

5. Conclusions

The two-point correlation function for the Dirichlet L -function with a primitive character is as follows

$$R_2(\epsilon) = \bar{d}^2(E) + R_2^{(\text{diag})}(\epsilon) + R_2^{(\text{off})}(\epsilon). \tag{54}$$

Here $\bar{d}(E)$ is the mean density of zeros

$$\bar{d}(E) = \frac{1}{2\pi} \ln\left(\frac{kE}{2\pi}\right). \tag{55}$$

The diagonal part of the correlation function, $R_2^{(\text{diag})}(\epsilon)$, and the off-diagonal part, $R_2^{(\text{off})}(\epsilon)$, have the form

$$R_2^{(\text{diag})}(\epsilon) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon^2} \ln \left[|\zeta(1+i\epsilon)|^2 \Phi^{(\text{diag})}(\epsilon) \Psi^{(\text{diag})}(\epsilon, k) \right] \tag{56}$$

and

$$R_2^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} e^{2\pi i \epsilon \bar{d}(E)} |\zeta(1+i\epsilon)|^2 \Phi^{(\text{off})}(\epsilon) \Psi^{(\text{off})}(\epsilon, k) + \text{c.c.}, \tag{57}$$

where the functions $\Phi^{(\text{diag})}(\epsilon)$ and $\Phi^{(\text{off})}(\epsilon)$ are the same as for the Riemann case (cf. (4) and (6)). The difference is in the functions $\Psi^{(\text{diag})}(\epsilon, k)$ and $\Psi^{(\text{off})}(\epsilon, k)$ given in (33) and (53) as a finite product of primes which divide the modulus k of the primitive character used to construct the L -function. As all these additional factors tend to 1 when $\epsilon \rightarrow 0$ the universal behavior of the two-point function for the Dirichlet zeros is the same as for the GUE/CUE.

Acknowledgments

JPK was supported by a grant from the Leverhulme Trust and by the Air Force Office of Scientific Research, Air Force Material Command, USAF, under grant number FA8655-10-1-3088. The US Government is authorized to reproduce and distribute reprints for Governmental purpose notwithstanding any copyright notation thereon.

Appendix. Contribution of the $r = 0$ term

For completeness let us consider separately the term with $t = 0$ in (48). Then

$$g(0) = \sum_{t_1=0}^{k-1} \chi(t_1) \bar{\chi}(t_1) = \phi(k) \tag{A.1}$$

and (48) takes the form

$$R_2^{(\text{off})}(\epsilon) = \frac{S(k)\phi(k)}{4\pi^2 k} e^{i\epsilon \ln(Ek/2\pi)} \sum_{(q,k)=1} \sum_{(R,q)=1} \left(\frac{q}{R}\right)^{1+i\epsilon} + \text{c.c.} \tag{A.2}$$

This differs from the Riemann case by the factor

$$\frac{S(k)\phi(k)}{k} = \frac{k}{\phi(k)} = \prod_{p|k} \frac{1}{1-p^{-1}}, \tag{A.3}$$

and by the restriction that the summation is performed not over all q but only over q coprime with k .

As in (50) we get

$$\sum_{(R,q)=1} \frac{1}{R^{1+i\epsilon}} = \zeta(1+i\epsilon) \prod_{p|k} \left(1 - \frac{1}{p^{1+i\epsilon}}\right) \tag{A.4}$$

and

$$\begin{aligned} \sum_{(q,k)=1} q^{1+i\epsilon} \left(\frac{\mu(q)}{\phi(q)}\right)^2 \prod_{p|q} \left(1 - \frac{1}{p^{1+i\epsilon}}\right) &= \prod_{(p,k)=1} \left(1 + \frac{1}{p^{1-i\epsilon}} \frac{1}{(1-1/p)^2} \left(1 - \frac{1}{p^{1+i\epsilon}}\right)\right) \\ &= \prod_{(p,k)=1} \frac{1}{1-p^{-1+i\epsilon}} \left(1 - \frac{(p^{i\epsilon}-1)^2}{(p-1)^2}\right) \\ &= \zeta(1-i\epsilon) \Phi^{(\text{off})}(\epsilon) \prod_{p|k} (1-p^{-1+i\epsilon}). \end{aligned} \tag{A.5}$$

Finally we obtain that the contribution of the term with $r = 0$ is

$$R_{2,0}^{(\text{off})}(\epsilon) = \frac{1}{4\pi^2} e^{i\epsilon \ln(Ek/2\pi)} |\zeta(1+i\epsilon)|^2 \Phi^{(\text{off})}(\epsilon) \Psi^{(\text{off})}(\epsilon, k) \prod_{p|k} \frac{p-p^{-i\epsilon}}{p-1} + \text{c.c.} \tag{A.6}$$

This differs from (52) only by the factor $\prod_{p|k} (p-p^{-i\epsilon})/(p-1)$ which tends to one when $\epsilon \rightarrow 0$ and, therefore, $R_{2,0}^{(\text{off})}(\epsilon)$ has exactly the correct singularity when $\epsilon \rightarrow 0$. It means that the term with $r = 0$ is responsible for the random matrix limit exactly as was shown in [11] by another method.

References

- [1] Montgomery H 1973 The pair correlation of the zeta function *Proc. Symp. Pure Math. (St. Louis, MO, 1972)* vol 24 (Providence, RI: American Mathematical Society) pp 181–93
- [2] Odlyzko A M 1989 The 10^{20} th zero of the Riemann zeta function and 70 million of its neighbours, unpublished
- [3] Keating J P and Snaith N C 2003 Random matrices and L -functions *J. Phys. A: Math. Gen.* **36** 2859
- [4] Hardy G H and Littlewood J E 1923 Some problems of ‘Partitio Numerorum’: III. On the expression of a number as a sum of primes *Acta Math.* **44** 1
- [5] Keating J P 1993 Quantum chaology and the Riemann zeta-function *Quantum Chaos* ed G Casati, I Guarneri and U Smilansky (Amsterdam: North-Holland) p 145
- [6] Bogomolny E B and Keating J P 1995 Random matrix theory and the Riemann zeros: I. Three- and four-point correlations *Nonlinearity* **8** 1115
Bogomolny E B and Keating J P 1996 Random matrix theory and the Riemann zeros: II. n -point correlations *Nonlinearity* **9** 911
- [7] Bogomolny E B and Keating J P 1996 Gutzwiller’s trace formula and spectral statistics: beyond the diagonal approximation *Phys. Rev. Lett.* **77** 1472
- [8] Bogomolny E 1999 Spectral statistics and periodic orbits *Proc. Int. School of Physics ‘Enrico Fermi’* (Varenna) p 333
- [9] Berry M V and Keating J P 1999 The Riemann zeros and eigenvalue asymptotics *SIAM Rev.* **41** 236
- [10] Bogomolny E 2007 Riemann zeta function and quantum chaos *Proc. Int. Conf. on Quantum Mechanics and Chaos (Osaka University, 2006)*; *Prog. Theor. Phys. Suppl.* **166** 19
- [11] Bogomolny E and Leboeuf P 1994 Statistical properties of the zeros of zeta functions—beyond the Riemann case *Nonlinearity* **7** 1155
- [12] Conrey J B, Farmer D W, Keating J P, Rubinstein M O and Snaith N C 2005 Integral moments of L -functions *Proc. Lond. Math. Soc.* **91** 33
- [13] Conrey J B, Farmer D W and Zirnbauer M R 2008 Autocorrelation of ratios of L -functions *Commun. Number Theory Phys.* **2** 593
- [14] Conrey J B and Snaith N C 2007 Applications of the L -functions ratios conjectures *Proc. Lond. Math. Soc.* **94** 594
- [15] Hardy G H 1921 Note on Ramanujan’s trigonometric function $c_q(n)$ and certain series of arithmetical functions *Proc. Camb. Phil. Soc.* **20** 263
- [16] Davenport H 2000 *Multiplicative Number Theory* 3rd edn (Berlin: Springer)