Large deviations of the top eigenvalue of random matrices and applications in statistical physics

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S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014), arXiv:1311.0580

Large spectrum of applications of random matrix theory

- Physics: nuclear physics, quantum chaos, disordered systems, mesoscopic transport, quantum entanglement, neural networks, gauge theory, string theory, cosmology, statistical physics (growth models, interface, directed polymers),...
- Mathematics: number theory, combinatorics, knot theory, determinantal point processes, integrable systems, free probability,...
- Statistics: multivariate statistics, principal component analysis (PCA), image processing, data compression, Bayesian model selection,...
- Information theory: signal processing, wireless communications,...
- Biology: sequence matching, RNA folding, gene expression networks,...
- Economy and finance: time series and big data analysis,...

«The Oxford handbook of random matrix theory», Ed. by G. Akemann, J. Baik and P. Di Francesco (2011)

Spectral statistics in random matrix theory (RMT)

- Basic model: real, symmetric, $N \times N$ Gaussian random matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,N} \\ M_{1,2} & M_{2,2} & \dots & M_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ M_{1,N} & M_{2,N} & \dots & M_{N,N} \end{pmatrix}$$

$$P(M) \propto \exp\left[-\frac{N}{2}\sum_{i,j} M_{i,j}^2\right]$$

$$\propto \exp\left[-\frac{N}{2}\mathrm{Tr}(M^2)\right]$$

$\Longrightarrow$ Invariant under rotation

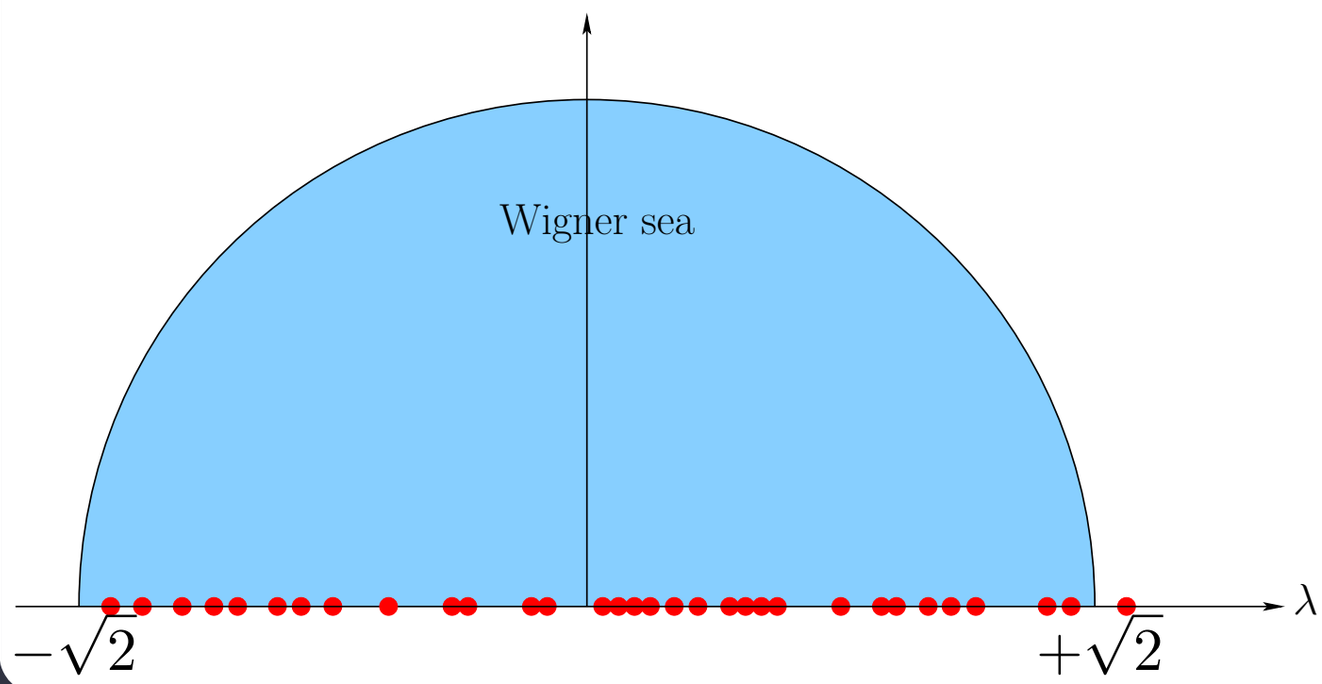
Gaussian orthogonal ensemble (GOE)

- The matrix M has N real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ which are strongly correlated

- Spectral statistics in RMT: statistics of $\lambda_1, \lambda_2, \dots, \lambda_N$

Largest (top) eigenvalue of random matrices

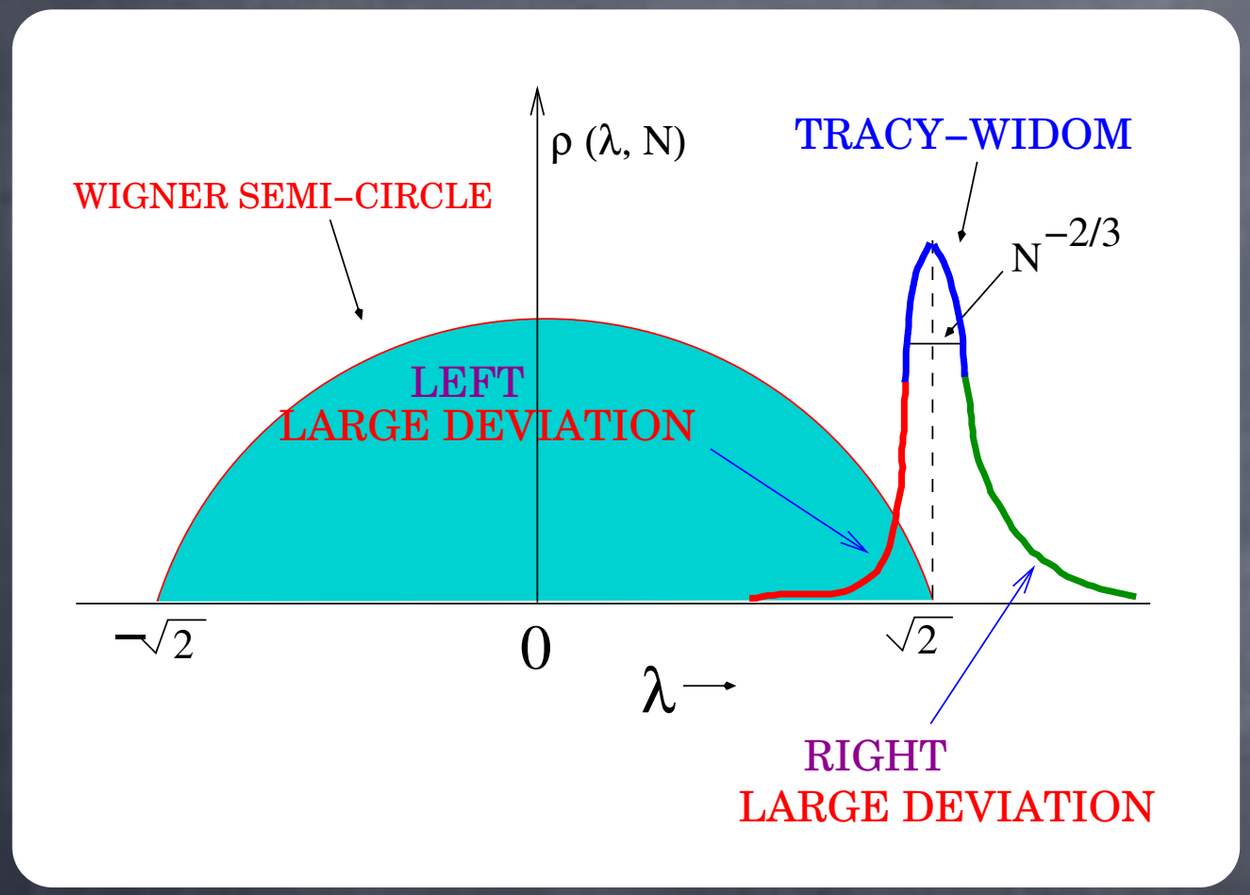
Density of eigenvalues for $N \gg 1$



Largest (top) eigenvalue of random matrices

- Recent excitements in statistical physics and mathematics on

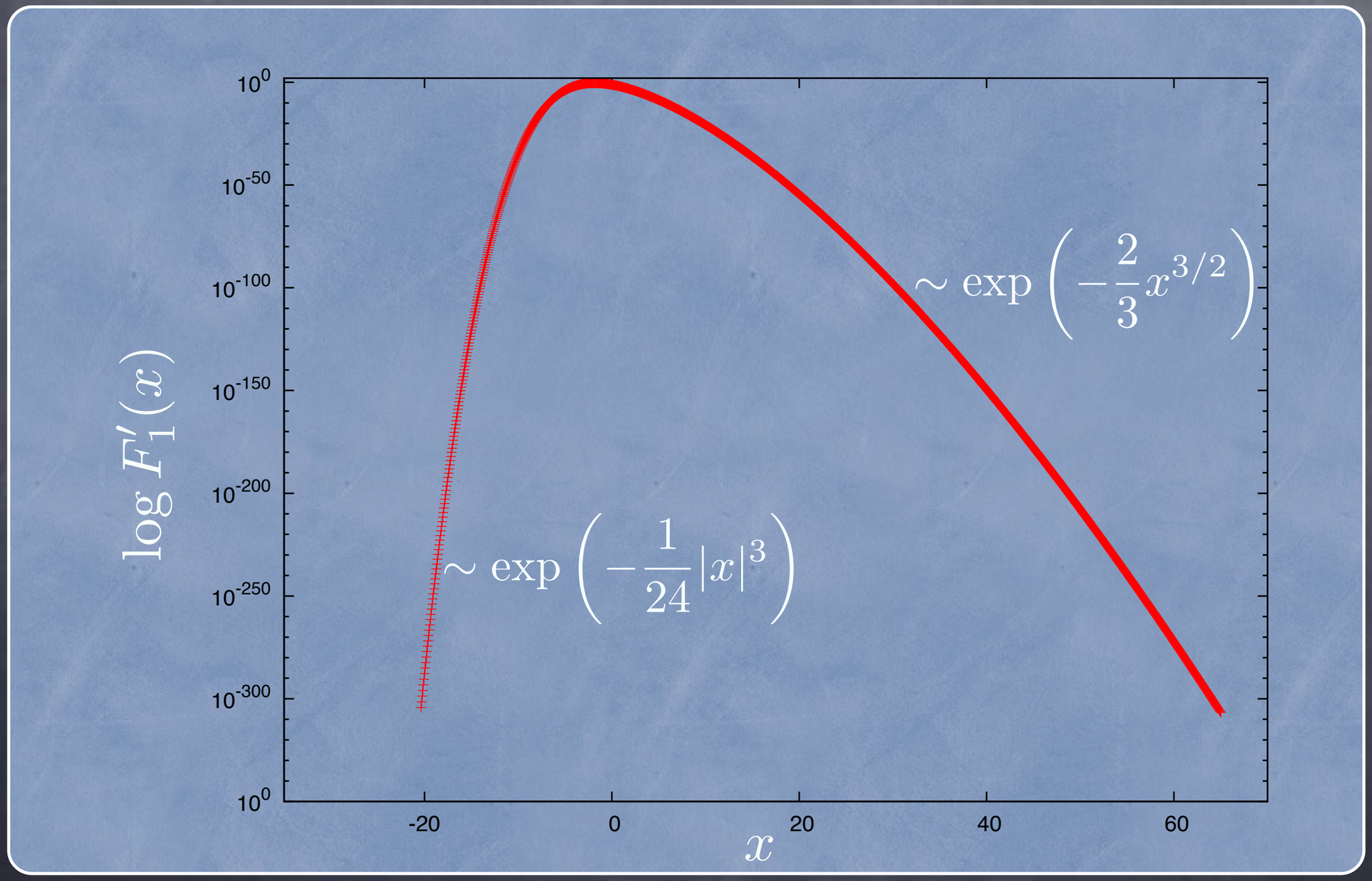
$$\lambda_{\max} = \max_{1 \le i \le N} \lambda_i \; : \; \text{largest eigenvalue}$$



Typical fluctuations (small):

✓ Tracy-Widom distribution

Tracy-Widom distribution



Largest (top) eigenvalue of random matrices

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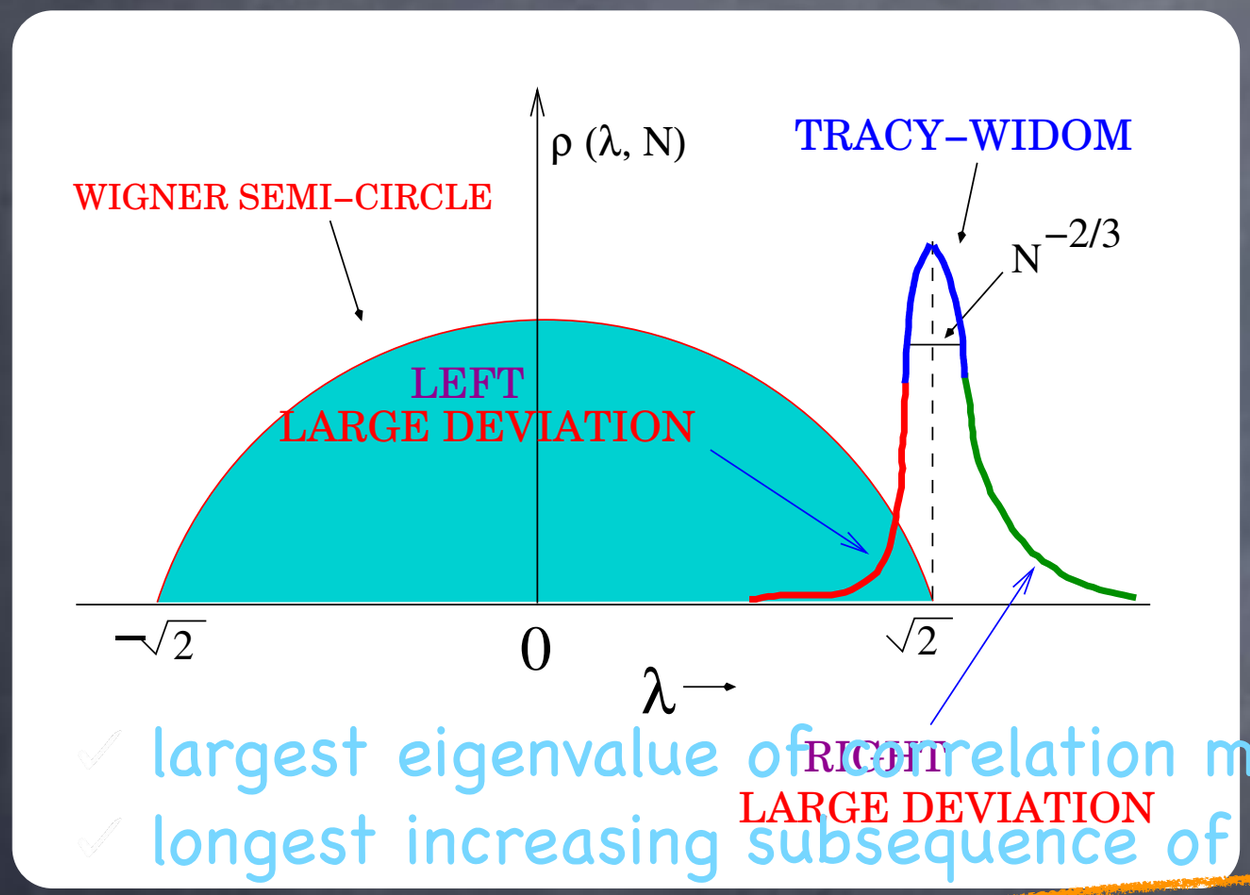


Typical fluctuations (small):

✓ Tracy-Widom distribution

✓ ubiquitous

✓ largest eigenvalue of correlation matrices (Wishart-Laguerre)

✓ longest increasing subsequence of random permutations

✓ directed polymers and growth models in the KPZ universality class

✓ continuum KPZ equation

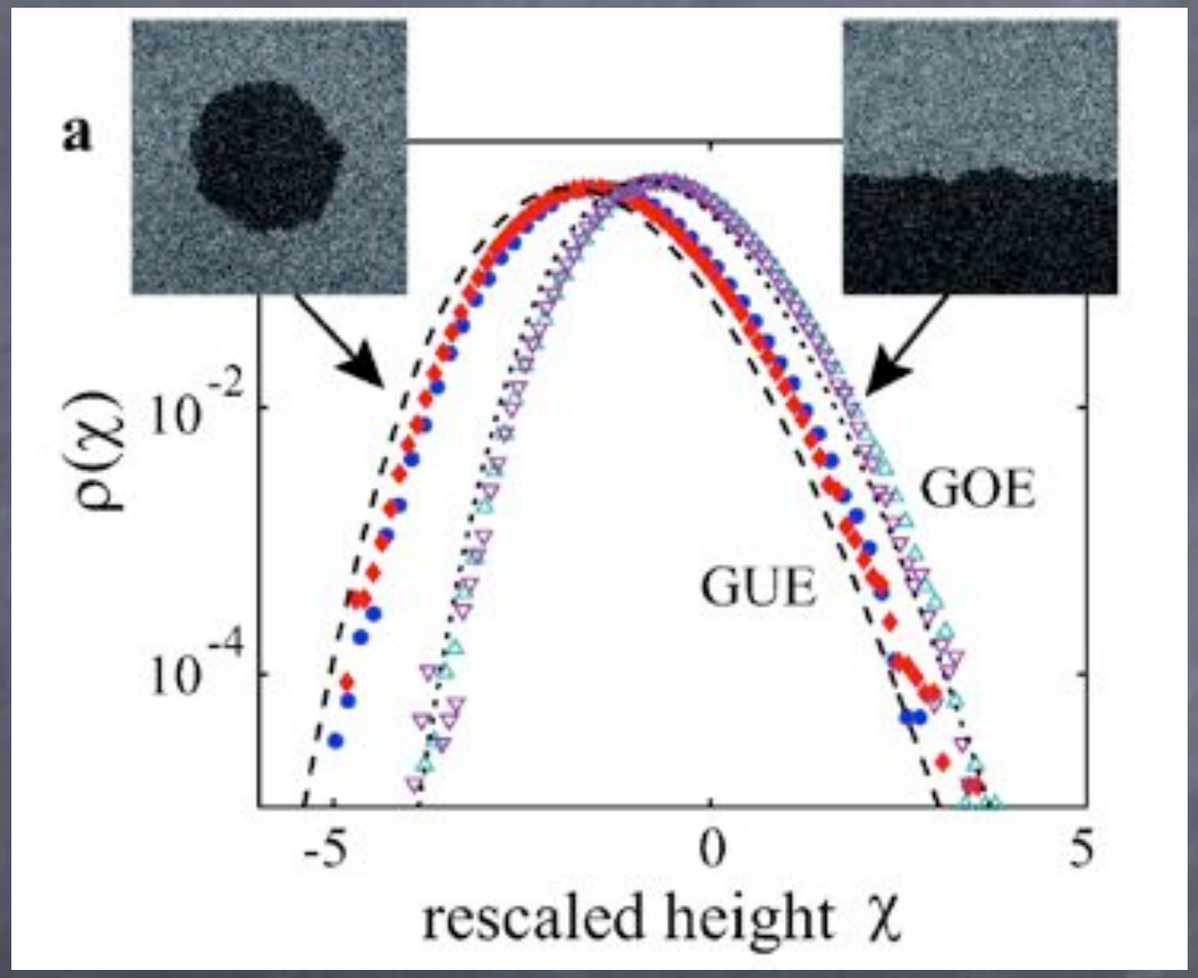
✓ sequence alignment problems

✓ mesoscopic fluctuations in quantum dots

✓ high-energy physics (Yang-Mills theory)...

Ubiquity of Tracy-Widom distributions

- Experimental observation of TW distributions for GOE $(\beta = 1)$ and GUE $(\beta = 2)$ in liquid crystals experiments (Carr-Helfrich instability)



Takeuchi & Sano '10

Takeuchi, Sano, Sasamoto & Spohn '11

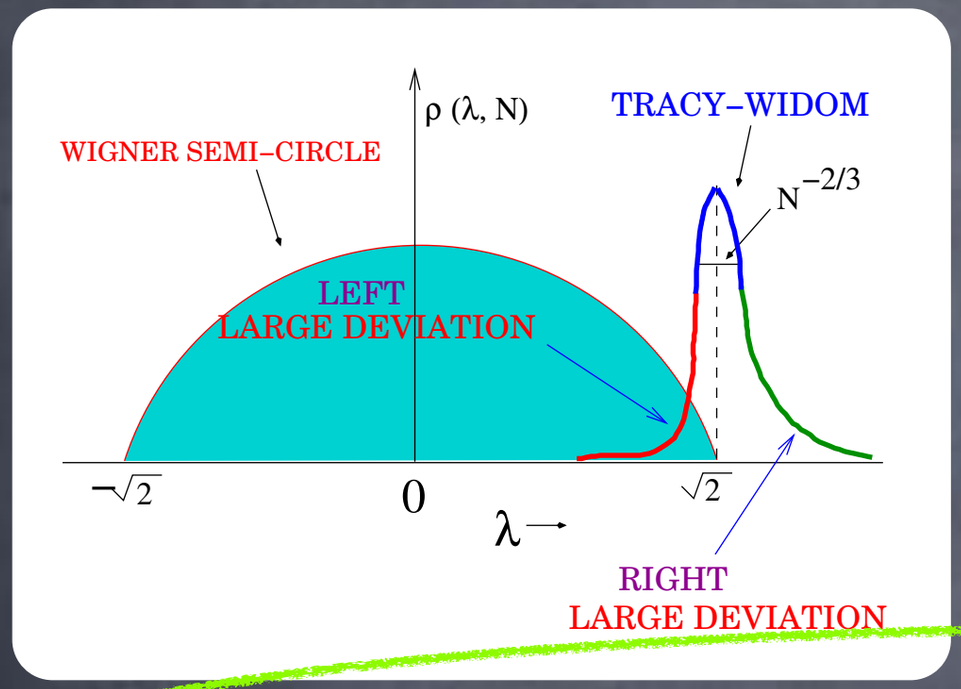
from Takeuchi, Sano, Sasamoto & Spohn, Sci. Rep. (Nature) 1, 34 (2011)

Q: universality of the Tracy-Widom distributions ?

Largest (top) eigenvalue of random matrices

- Recent excitements in statistical physics and mathematics on

$$\lambda_{\max} = \max_{1 \le i \le N} \lambda_i \; : \; \text{largest eigenvalue}$$



Typical fluctuations (small):

✓ Tracy-Widom distribution

✓ ubiquitous

Q: universality of the Tracy-Widom distributions ?

- In this talk: atypical and large fluctuations of $\lambda_{\max}$

✓ Large deviation functions

✓ Third order phase transition

Stability of a large complex system

Will a Large Complex System be Stable?

ROBERT M. MAY

NATURE VOL. 238 AUGUST 18 1972

- Stable non-interacting population of N species with equilibrium densites ρ_i^*

  Slightly perturbed densities $x_i(t) = \rho_i(t) - \rho_i^*$ evolve via

$$\frac{dx_i(t)}{dt} = -x_i(t)$$

(assuming identical damping times)

- Switch on interactions between the species

$$\frac{dx_i(t)}{dt} = -x_i(t) + \alpha \sum_{j=1}^{N} M_{i,j} x_j(t)$$

coupling strength

random interaction matrix

- Q: what is the proba. that the system remains stable once the interactions are switched on ?

Linear stability criterion

- Linear dynamical system

$$\frac{d\,\mathbf{x}(\mathbf{t})}{dt} = (\alpha \mathbf{M} - \mathbb{I})\mathbf{x}(\mathbf{t})$$

- Eigenvalues of $\mathbf{M}$: $\lambda_1, \lambda_2, \ldots, \lambda_N$

- The system is stable iff $\alpha\lambda_i < 1$, $\forall i = 1, 2, \ldots, N$

i.e. iff $\lambda_{\max} < \dfrac{1}{\alpha} = w$

- Proba. that the system is stable $P_{\text{stable}}(\alpha, N)$

$$P_{\text{stable}}(\alpha, N) = \text{Proba.}[\lambda_{\max} < 1/\alpha = w]$$

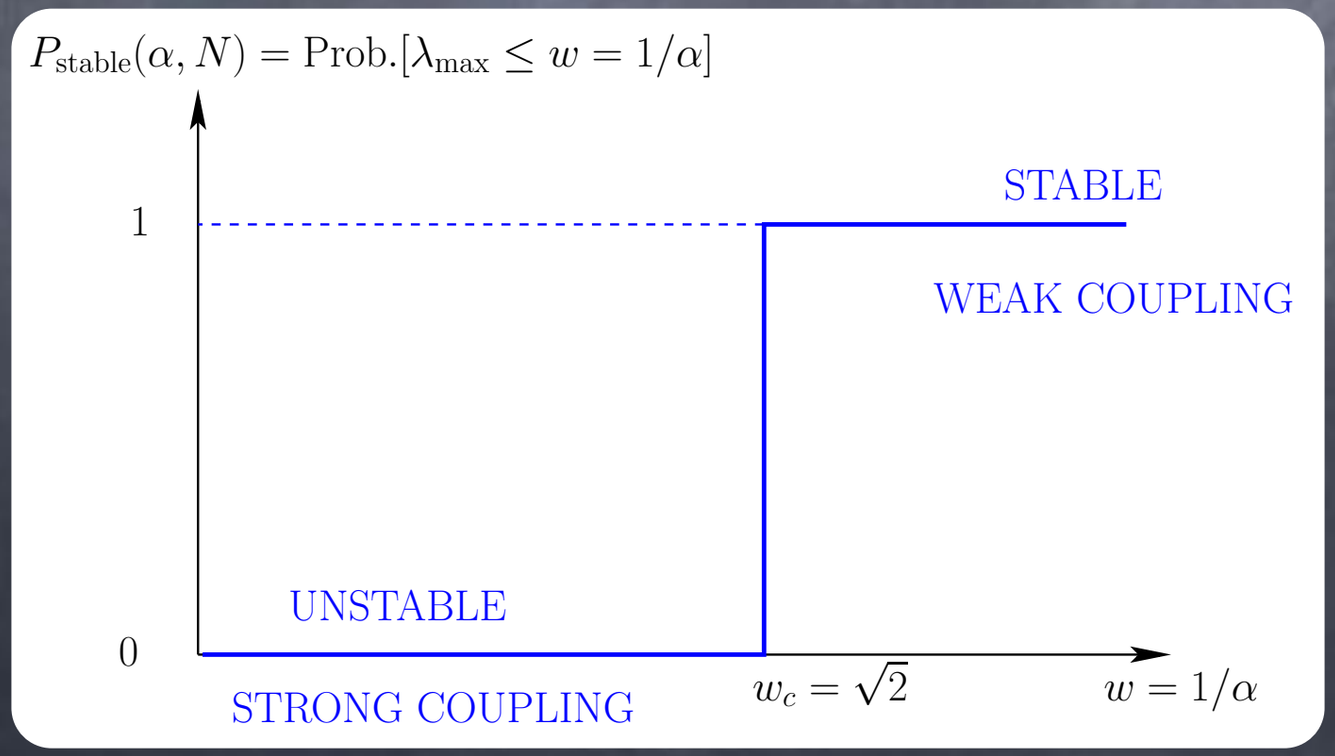
Stable/Unstable transition for large systems

- Assuming that the interaction matrix is real, symmetric and Gaussian

$$M_{i,j} = M_{j,i} \ , \ P(\mathbf{M})d\mathbf{M} \propto \exp\left[-\frac{N}{2}\sum_{i,j} M_{i,j}^2\right] d\mathbf{M} = \exp\left[-\frac{N}{2}\mathbf{Tr}(\mathbf{M}^2)\right] d\mathbf{M}$$

- May observed a sharp transition in the limit $N \to \infty$

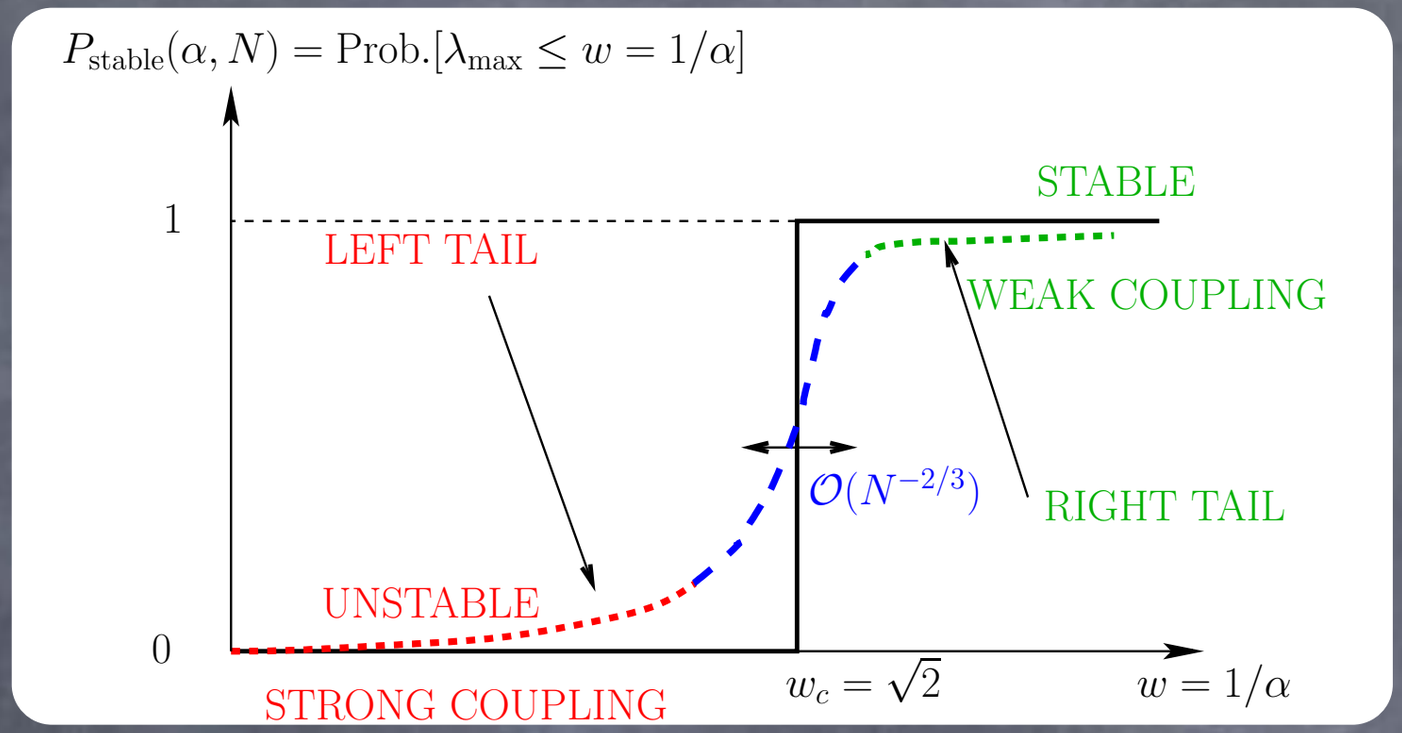
$$\lim_{N\to\infty} P_{\text{stable}}(\alpha, N) = \begin{cases} 1 \ , \ \alpha < \alpha_c = 1/\sqrt{2} & \text{: stable, weakly interacting phase} \\ 0 \ , \ \alpha > \alpha_c = 1/\sqrt{2} & \text{: unstable, strongly interacting phase} \end{cases}$$



May '72

Stable/Unstable transition for large systems

- What happens for finite but large systems, of size $N \gg 1$?



- ✓ Is there any thermodynamic sense to this transition ?
- ✓ What is the analogue of the free energy ?
- ✓ What is the order of this transition ?

Coulomb Gas approach

Gaussian random matrix models

- $N \times N$ random matrix : $\mathbf{M} \equiv M_{i,j}$

- Standard Dyson 's ensembles: Orthogonal, Unitary, Symplectic
  (GOE) (GUE) (GSE)

- Gaussian probability measure $P(\mathbf{M})d\mathbf{M} \propto \exp\left[-\frac{\beta}{2}N\mathrm{Tr}(\mathbf{M}^\dagger\mathbf{M})\right]d\mathbf{M}$

- Joint PDF of the real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ Wigner '51

$$P_{\text{joint}}(\lambda_1, \lambda_2, \cdots, \lambda_N) = \frac{1}{Z_N}\exp\left[-\frac{\beta}{2}N\sum_{i=1}^{N}\lambda_i^2\right]\prod_{i<j}|\lambda_i - \lambda_j|^\beta$$

with $\beta = 1$ (GOE), $\beta = 2$ (GUE) and $\beta = 4$ (GSE)

- Partition function

$$Z_N = \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \exp\left[-\frac{\beta}{2}N\sum_{i=1}^{N}\lambda_i^2\right]\prod_{i<j}|\lambda_i - \lambda_j|^\beta$$

Coulomb Gas picture and Wigner semi-circle

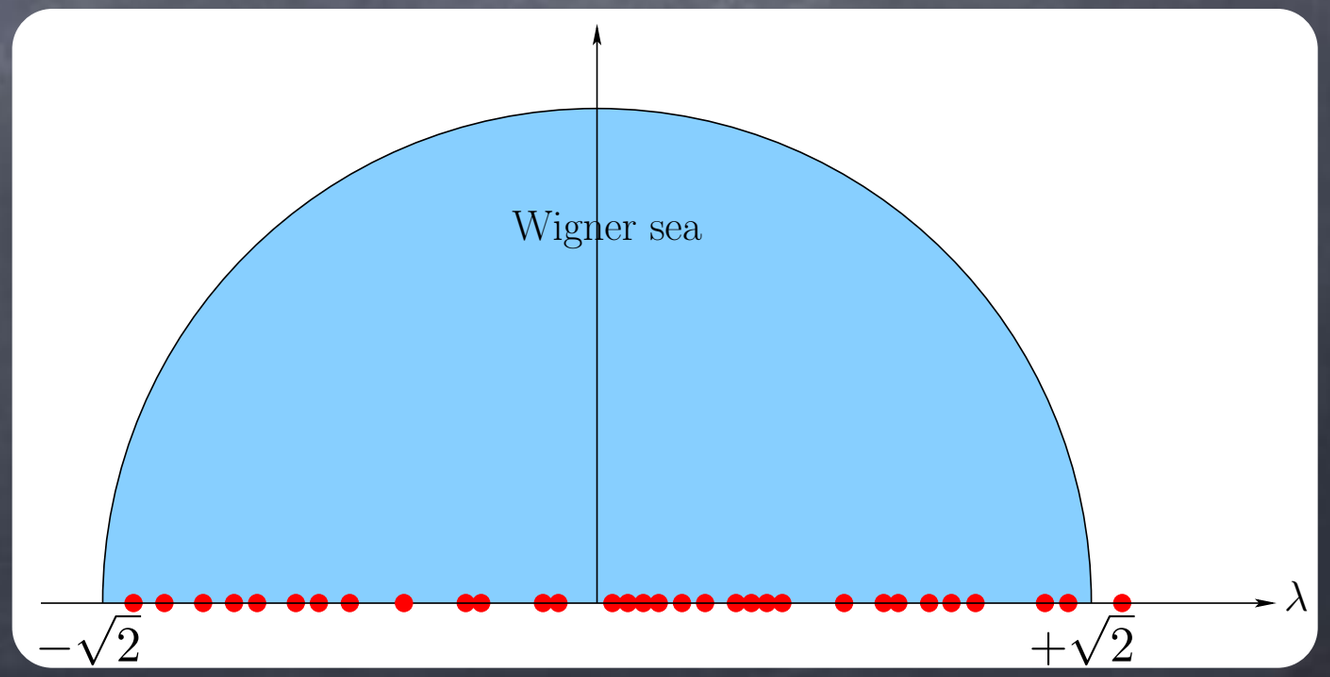
- Rewrite the partition function as

$$Z_N(\beta) = \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \exp\left[-\beta\left(\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2}\sum_{i\neq j}\ln|\lambda_i - \lambda_j|\right)\right]$$

- 2-d Coulomb gas confined to a line, with β the inverse temperature
- Typical scale of the eigenvalues: $N^2\lambda_{\rm typ}^2 \sim N^2 \Longrightarrow \lambda_{\rm typ} \sim \mathcal{O}(1)$
- Mean density of eigenvalues

$$\rho_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(\lambda - \lambda_i)\rangle$$

$$\xrightarrow[N\to\infty]{} \rho_{\rm SC}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$



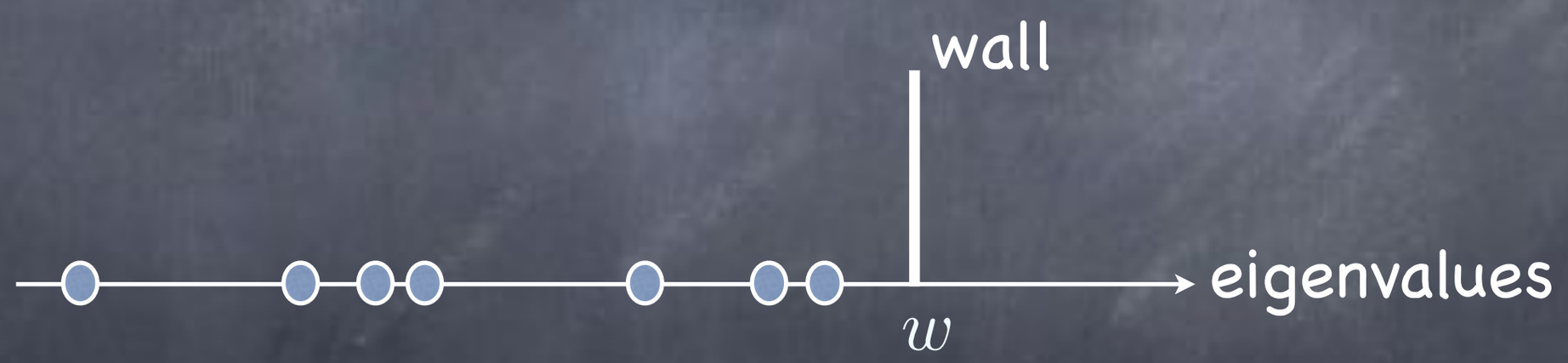
Coulomb Gas with a wall

Cumulative distribution function of $\lambda_{\max}$

$$\Pr.[\lambda_{\max} \leq w] = \Pr.[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)}$$

$$\boxed{Z_N(w) = \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp\left[-\beta N^2 E[\{\lambda_i\}]\right]}$$

$$E[\{\lambda_i\}] = \frac{1}{2N}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2N^2}\sum_{j\neq k}\log|\lambda_j - \lambda_k|$$



What happens when the wall is moved ?

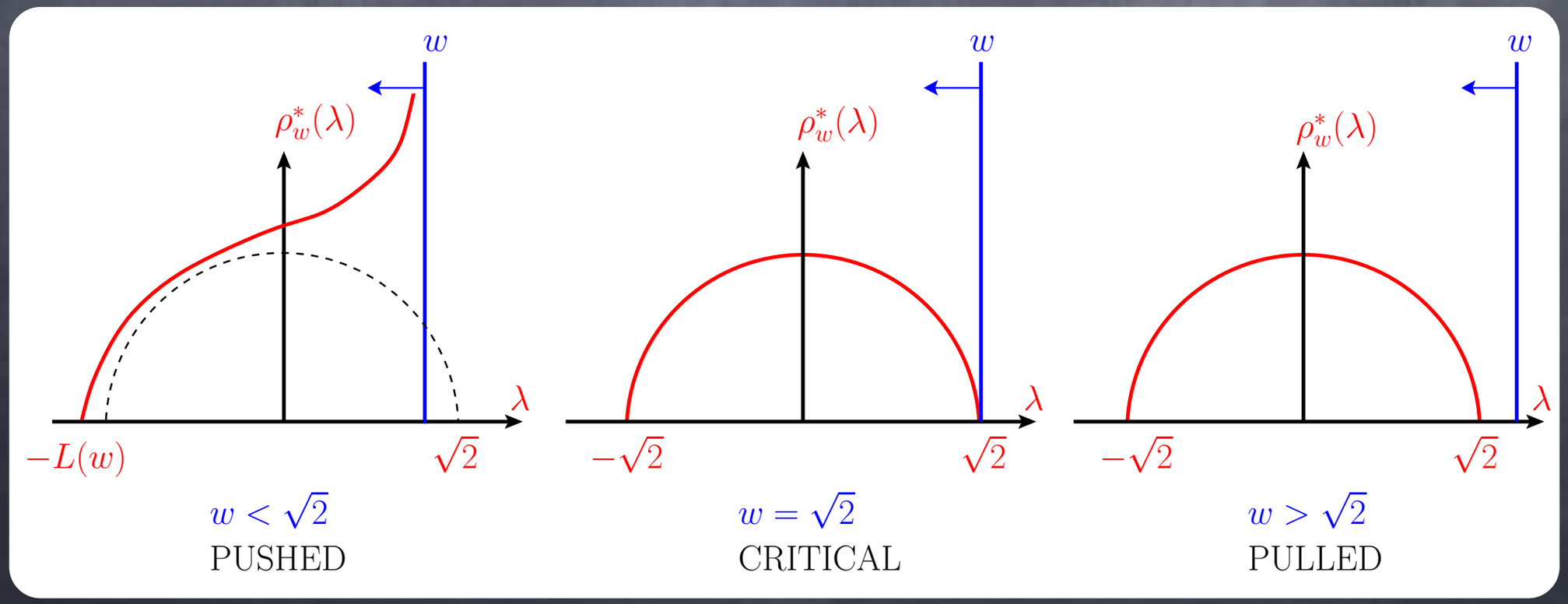
Pushed vs. pulled Coulomb gas

$$Z_N(w) = \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp\left[-\beta N^2 E[\{\lambda_i\}]\right]$$

$$E[\{\lambda_i\}] = \frac{1}{2N}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2N^2}\sum_{j\neq k}\log|\lambda_j - \lambda_k|$$

- Saddle point analysis Dean & Majumdar '06, '08

Mean density of eigenvalues in presence of the wall $\rho_w^*(\lambda)$



Left large deviation function

$$F(w,N) = \Pr.[\lambda_{\max} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim \exp[-\beta N^2 \Phi_-(w)] \ , \ w < \sqrt{2}$$

i.e. $$\lim_{N\to\infty} -\frac{1}{\beta N^2} F(w,N) = \Phi_-(w)$$

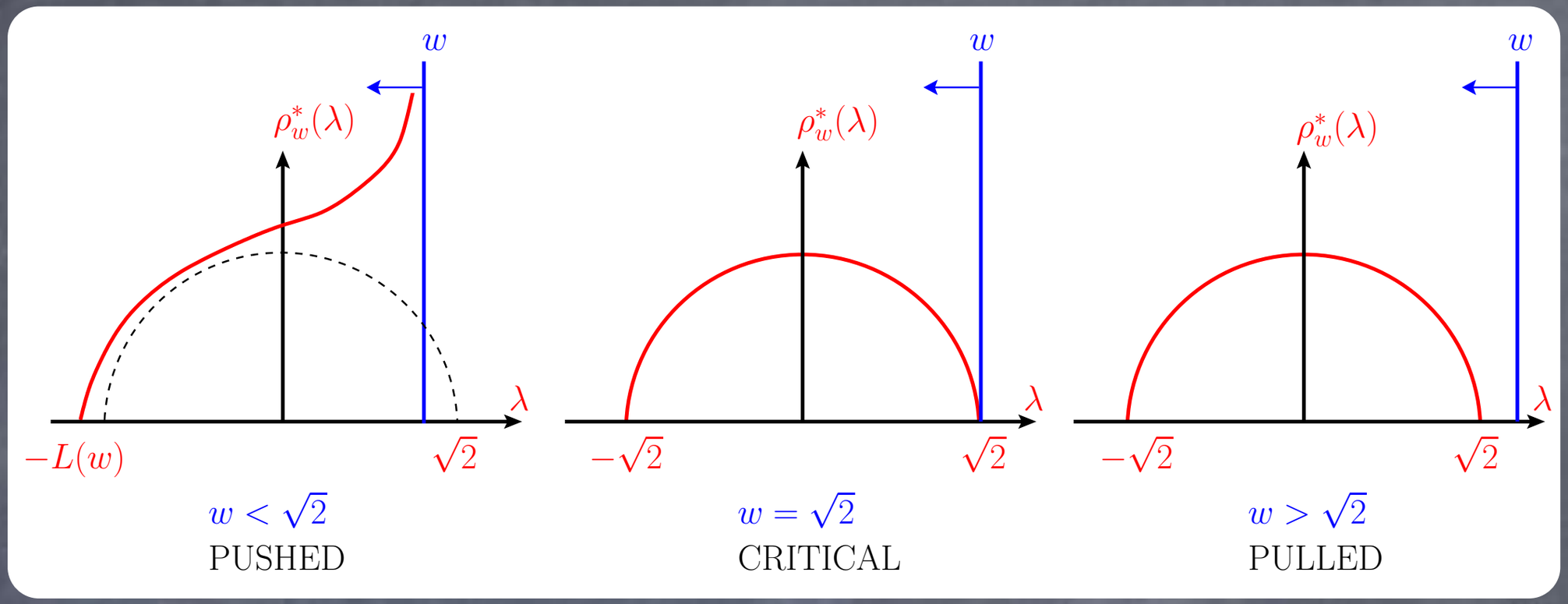
- Physically, $N^2\Phi_-(w)$ is the energy to push the Coulomb gas

- Left deviation function $$\Phi_-(w) = \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln 18 - 2\ln\left(w + \sqrt{w^2+6}\right)\right)\Big] \ , \ w < \sqrt{2}$$

when $w \to \sqrt{2}^-$ $$\Phi_-(w) \sim \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3 \ , \ w \to \sqrt{2}^-$$

Dean & Majumdar '06, '08

Right large deviation function



- The saddle point equation yields a trivial result for $w > \sqrt{2}$

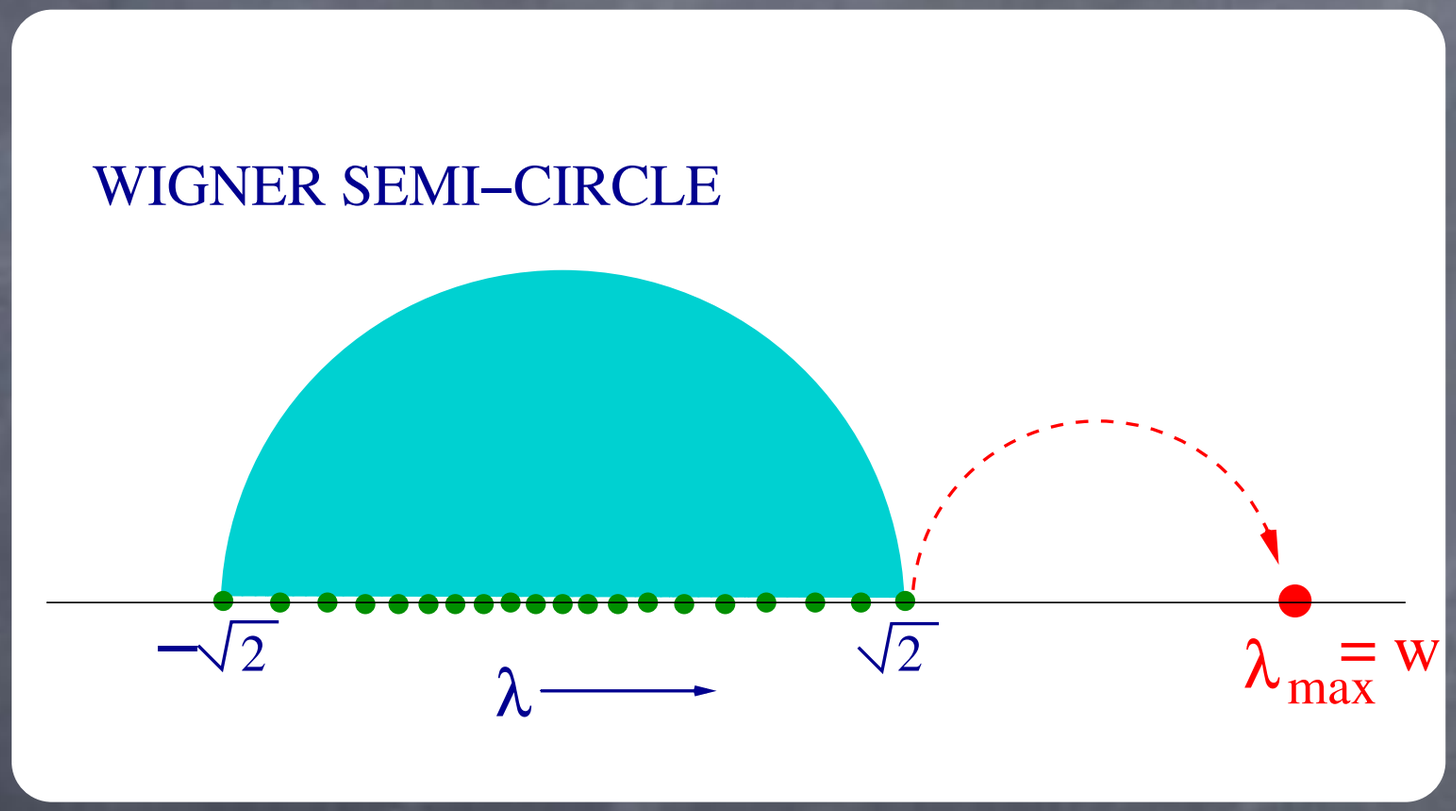
$$F(w, N) = \Pr.[\lambda_{\max} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim 1 \ , \ w > \sqrt{2}$$

$\Longrightarrow$ Non trivial corrections requires a different approach

Right tail: pulled Coulomb gas

$$F(w,N) = \Pr.[\lambda_{\max} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim 1 - \exp(-N\Phi_+(w)) \ , \ w > \sqrt{2}$$

$N\Phi_+(w)$: energy to pull a single charge out of the Wigner sea



$$\Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[\frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right]$$

Majumdar & Vergassola '09

when $w \to \sqrt{2}^+$ $\quad \Phi_+(w) \sim \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2} \ , \ w \to \sqrt{2}^+$

Third order phase transition

$$\Pr.[\lambda_{\max} < w] = F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right] & , w < \sqrt{2} \,\&\, |w-\sqrt{2}| \sim \mathcal{O}(1) \\ \mathcal{F}_\beta\left(\sqrt{2} N^{\frac{2}{3}}(w-\sqrt{2})\right) & , \quad |w-\sqrt{2}| \sim \mathcal{O}(N^{-\frac{2}{3}}) \\ 1-\exp\left[-\beta N \Phi_+(w)\right] & , w > \sqrt{2} \,\&\, |w-\sqrt{2}| \sim \mathcal{O}(1) \end{cases}$$

unstable

stable

- This implies the behavior of the free energy

$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} \Phi_-(w) \propto (\sqrt{2}-w)^3 \,, & w < \sqrt{2} \,, \\ 0 & w > \sqrt{2} \end{cases}$$

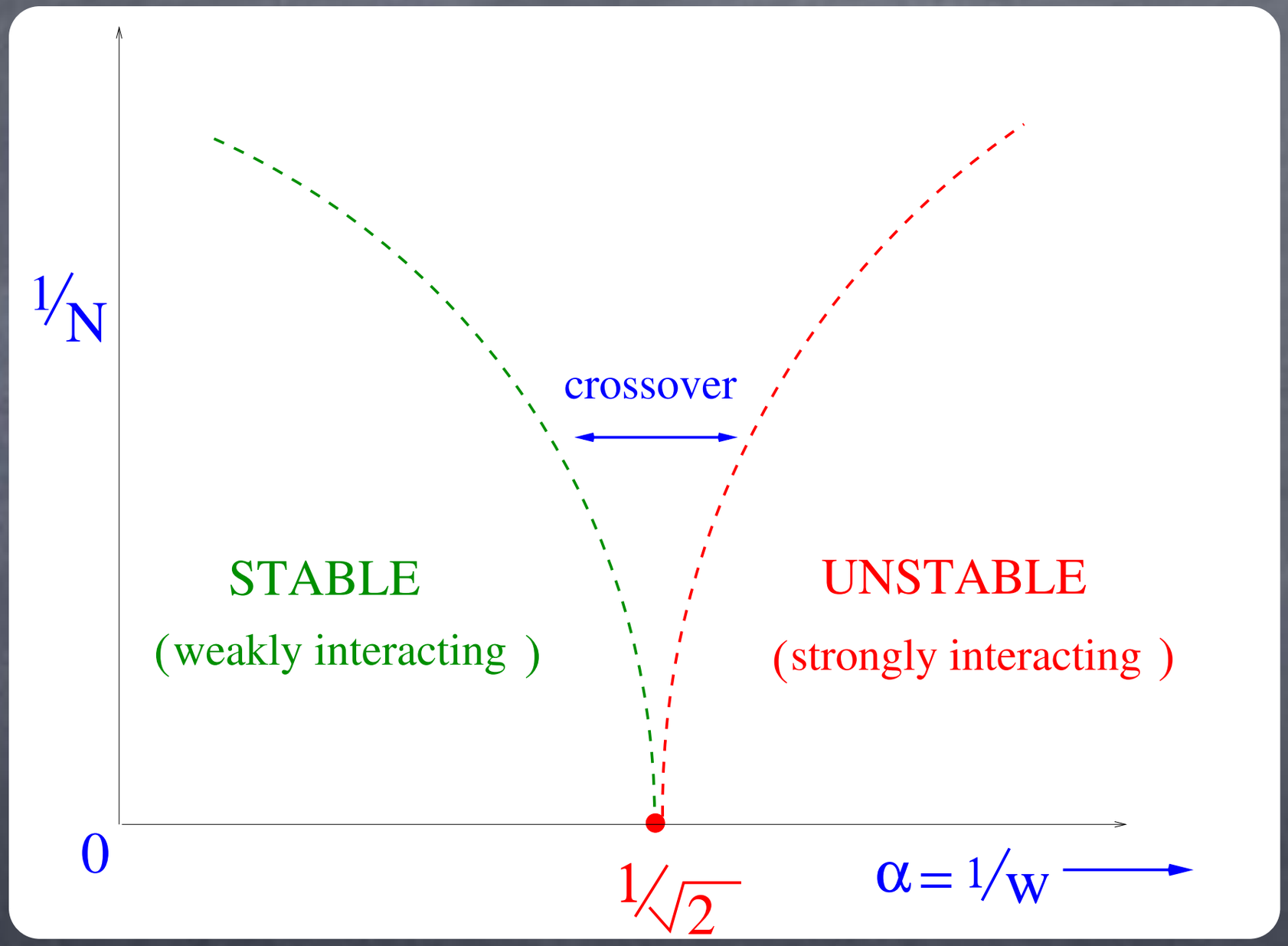
- The third derivative of the free energy is discontinuous

$\Longrightarrow$ Third order phase transition

- The crossover, for finite N, between the two phases is described by the Tracy-Widom distribution

S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014)

Third order phase transition



S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014)

Similar third order phase transition in gauge theory

PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

Possible third-order phase transition in the large-N lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 10 July 1979)

The large-N limit of the two-dimensional $U(N)$ (Wilson) lattice gauge theory is explicitly evaluated for all fixed $\lambda = g^2N$ by steepest-descent methods. The λ dependence is discussed and a third-order phase transition, at $\lambda = 2$, is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large-N four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

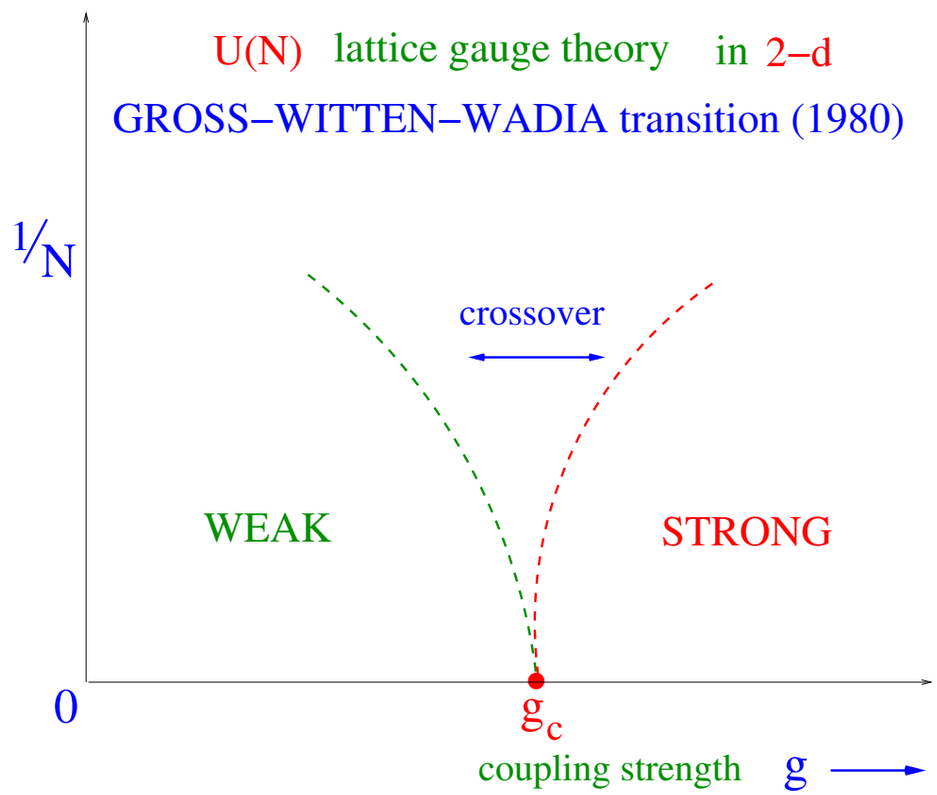
Volume 93B, number 4 PHYSICS LETTERS 30 June 1980

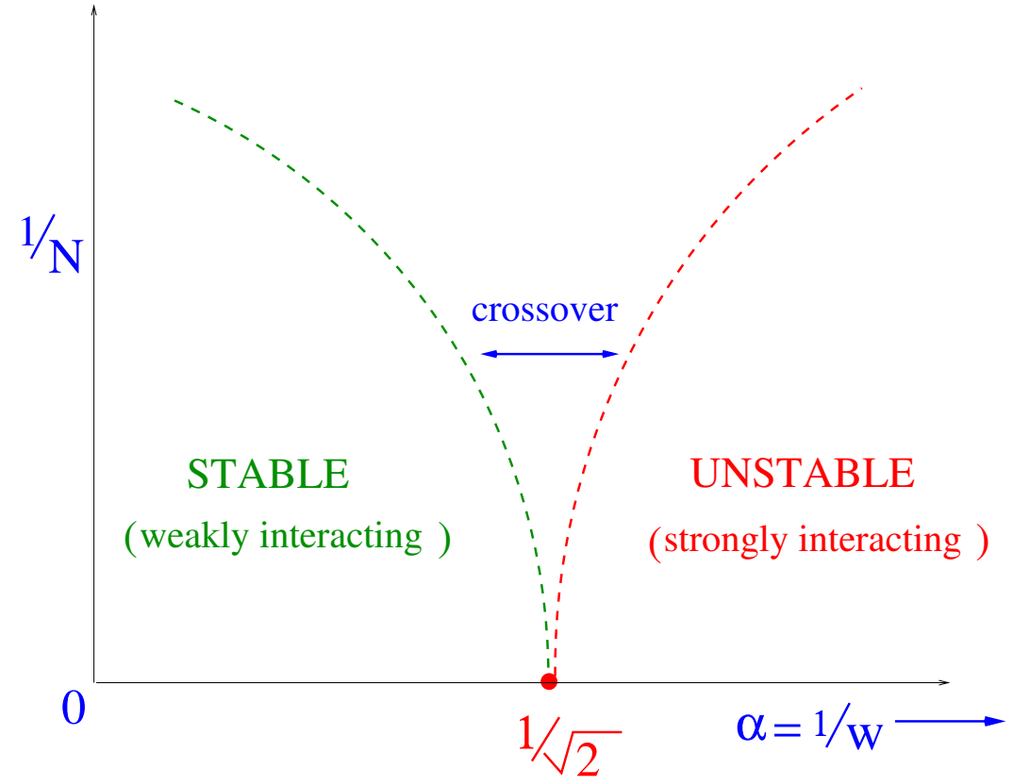
$N = \infty$ PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE MODEL LATTICE GAUGE THEORIES *

Spenta R. WADIA
The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Received 27 March 1980

Similar third order phase transition in gauge theory





Similar transition in $U(N)$ lattice gauge theory

Unstable phase = strong coupling phase of Yang-Mills gauge theory
Stable phase = weak coupling phase of Yang-Mills gauge theory

Tracy-Widom ditribution describes the crossover between the two regimes (at finite but large N): double scaling regime

Conclusion

- Largest eigenvalue $\lambda_{\max}$ of a Gaussian random matrix

  ✓ Application to the stability of large complex system

- Proba. distrib. func. (PDF) of $\lambda_{\max}$: Coulomb gas (CG) with a wall

  ✓ Tracy-Widom distribution

  ✓ Physics of large deviation tails Left tail: pushed CG
  Right tail: unpushed CG

  ⟹ Third order phase transition pushed/unpushed
  unstable/stable

- Similar third order transition in Yang-Mills gauge theories and other systems (conductance fluctuations, complexity in spin glasses, non-intersecting Brownian motions...)