Large deviations and phase transitions in random matrix theory and related topics

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LPTMS, CNRS-Université Paris-Sud XI

«Matrices aléatoires et applications»
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Collaborators:

- Alain Comtet (LPTMS, Orsay)
- Peter J. Forrester (Math. Dept., Univ. of Melbourne)
- Satya N. Majumdar (LPTMS, Orsay)

P. J. Forrester, S. N. Majumdar, G. S., Nucl. Phys. B844, 500 (2011)

G. S., A. Comtet, S. N. Majumdar, P. J. Forrester, J. Stat. Phys. 150, 491 (2013)

S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014)

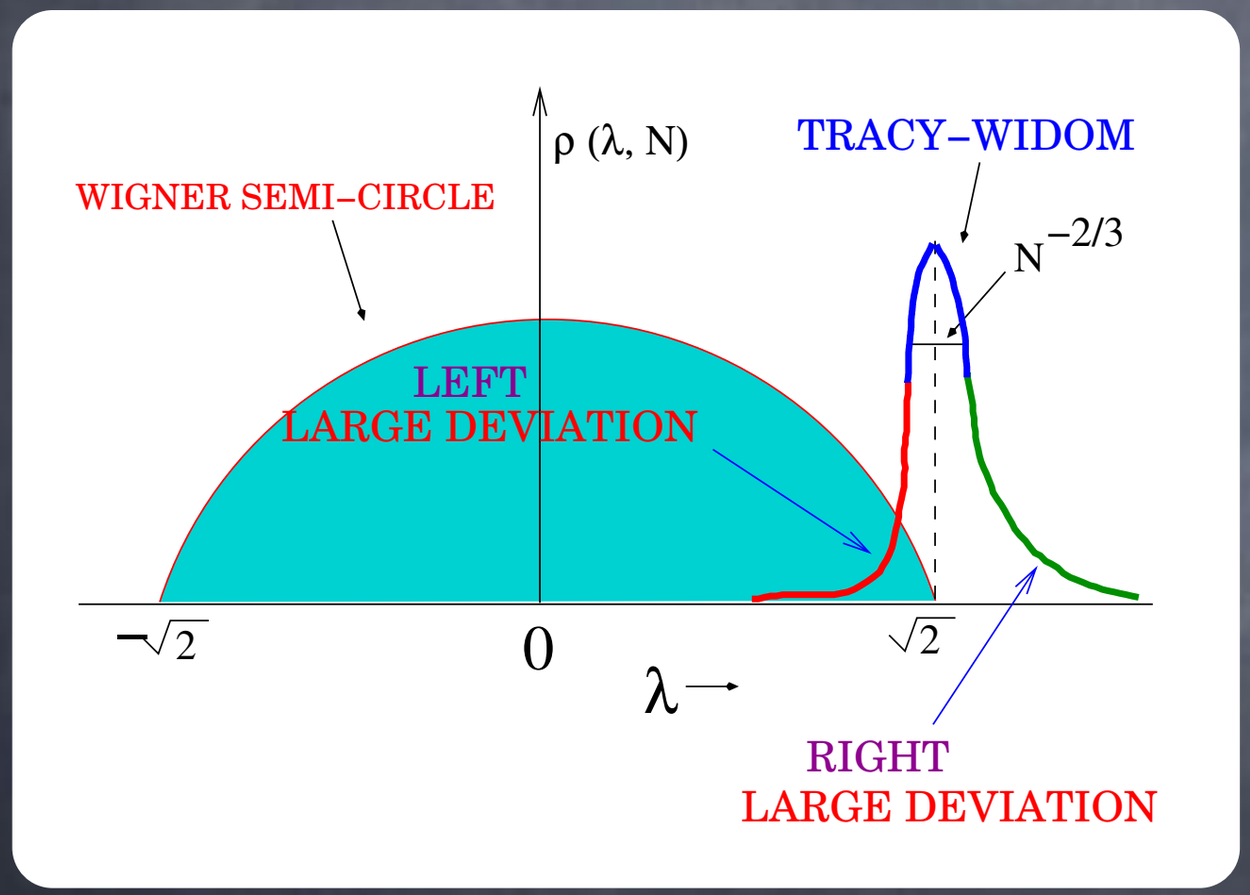
Main motivation: largest eigenvalue of random matrices

- Recent excitements in statistical physics and mathematics on

$$\lambda_{\max} = \max_{1 \le i \le N} \lambda_i \; : \; \text{largest eigenvalue of random matrices}$$

- Gaussian random matrices belonging to Dyson 's ensembles

GOE, GUE, GSE



Typical fluctuations (small):

✓ Tracy-Widom distributions

✓ ubiquitous/universal

KPZ equation, directed polymer, random permutation, correlation matrices, sequence alignment,...

- In this talk: atypical and large fluctuations of $\lambda_{\max}$

✓ Large deviation functions

✓ Third order phase transition

Outline

- Largest eigenvalue $\lambda_{\max}$ of a Gaussian random matrix
  - ✓ Application to the stability of large complex system
    - ⟹ Hint at a phase transition
- Proba. distrib. func. (PDF) of $\lambda_{\max}$: Coulomb gas (CG) (with a wall)
  - ✓ Tracy-Widom distribution
  - ✓ Physics of large deviation tails
    - ⟹ Third order phase transition
- Related third order transition in other systems
  - ✓ Non-intersecting Brownian motions
- Summary and conclusion

Stability of a large complex system

Will a Large Complex System be Stable?

ROBERT M. MAY

NATURE VOL. 238 AUGUST 18 1972

- Stable non-interacting population of N species with equilibrium densites ρ_i^*

  Slightly perturbed densities $x_i(t) = \rho_i(t) - \rho_i^*$ evolve via

$$\frac{dx_i(t)}{dt} = -x_i(t)$$

(assuming identical damping times)

- Switch on interactions between the species

$$\frac{dx_i(t)}{dt} = -x_i(t) + \alpha \sum_{j=1}^{N} M_{i,j} x_j(t)$$

coupling strength

random interaction matrix

- Q: what is the proba. that the system remains stable once the interactions are switched on ?

Linear stability criterion

- Linear dynamical system

$$\frac{d\,\mathbf{x}(\mathbf{t})}{dt} = (\alpha \mathbf{M} - \mathbb{I})\mathbf{x}(\mathbf{t})$$

- Eigenvalues of $\mathbf{M}$: $\lambda_1, \lambda_2, \ldots, \lambda_N$

- The system is stable iff $\alpha\lambda_i < 1$, $\forall i = 1, 2, \ldots, N$

i.e. iff $\lambda_{\max} < \dfrac{1}{\alpha} = w$

- Proba. that the system is stable $P_{\text{stable}}(\alpha, N)$

$$P_{\text{stable}}(\alpha, N) = \text{Proba.}[\lambda_{\max} < 1/\alpha = w]$$

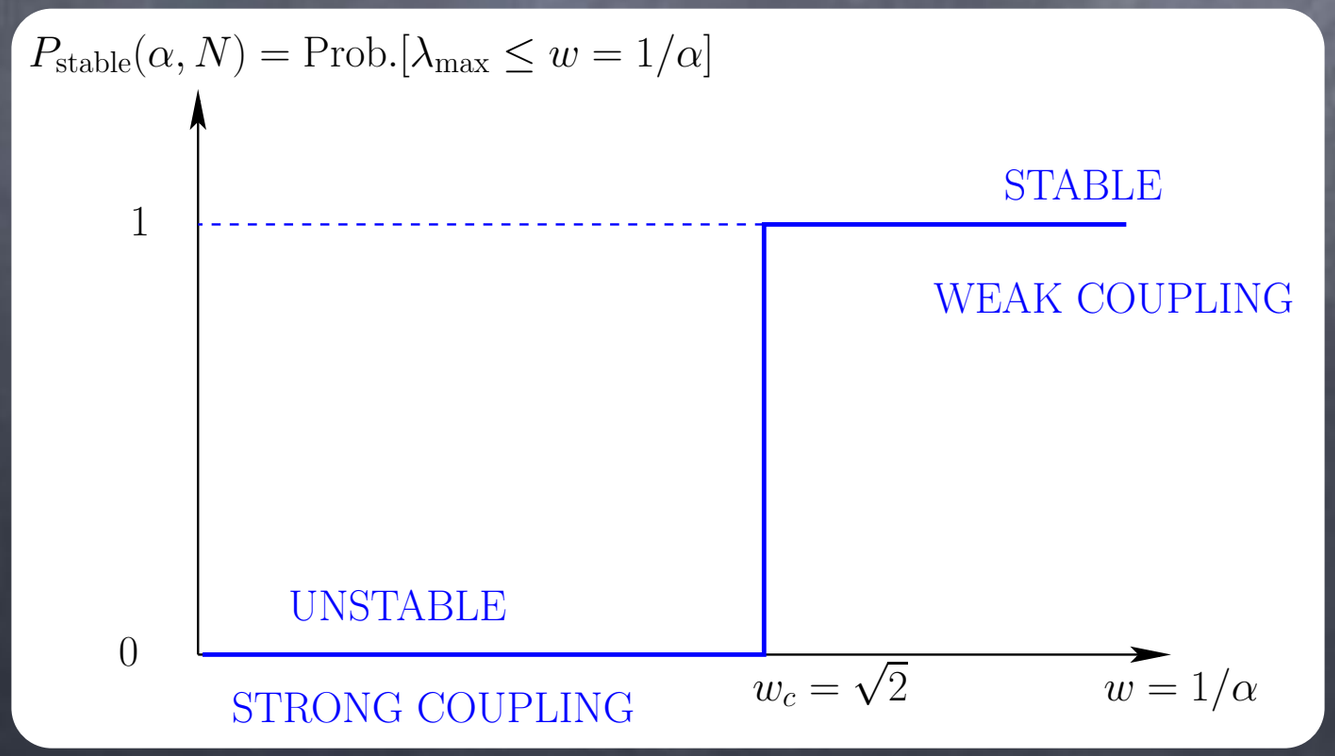
Stable/Unstable transition for large systems

- Assuming that the interaction matrix is real, symmetric and Gaussian

$$M_{i,j} = M_{j,i} \ , \ P(\mathbf{M})d\mathbf{M} \propto \exp\left[-\frac{N}{2}\sum_{i,j} M_{i,j}^2\right] d\mathbf{M} = \exp\left[-\frac{N}{2}\mathbf{Tr}(\mathbf{M}^2)\right] d\mathbf{M}$$

- May observed a sharp transition in the limit $N \to \infty$

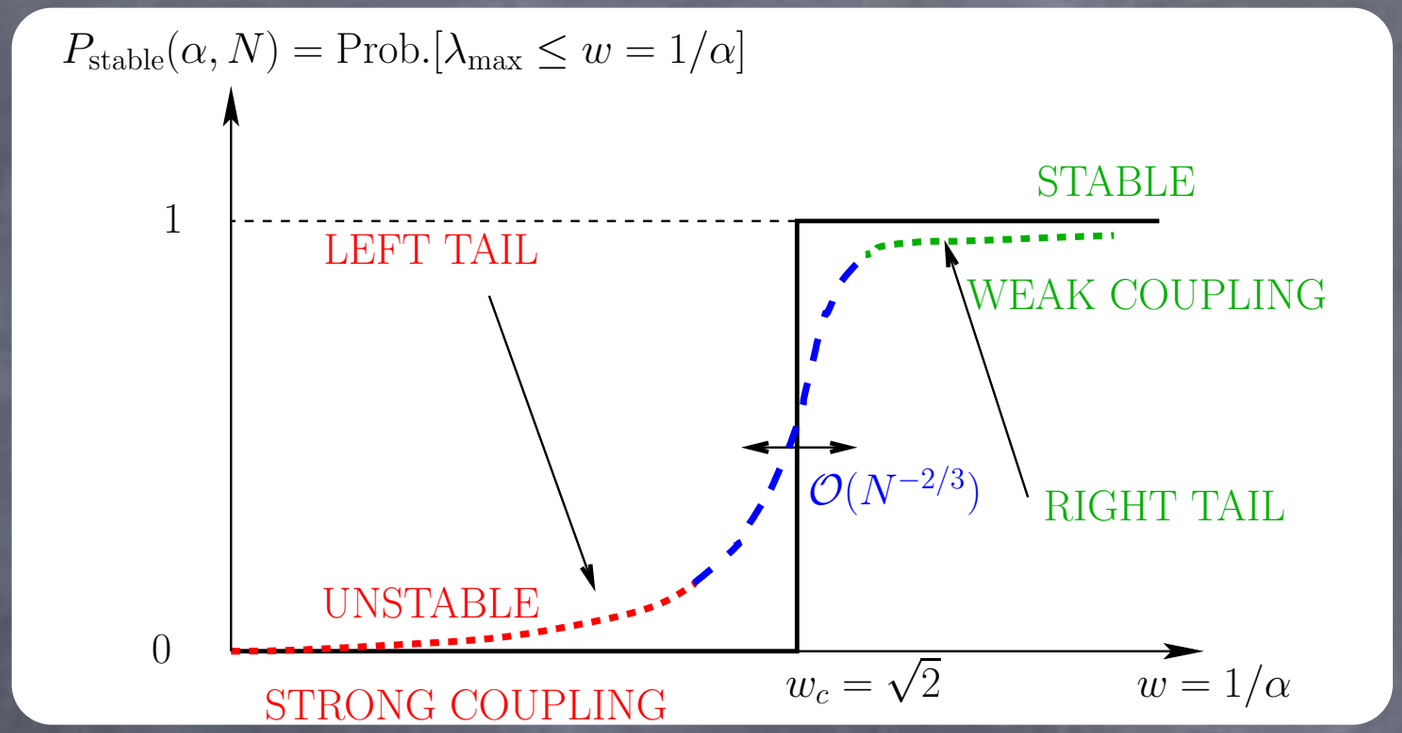
$$\lim_{N\to\infty} P_{\text{stable}}(\alpha, N) = \begin{cases} 1 \ , \ \alpha < \alpha_c = 1/\sqrt{2} & \text{: stable, weakly interacting phase} \\ 0 \ , \ \alpha > \alpha_c = 1/\sqrt{2} & \text{: unstable, strongly interacting phase} \end{cases}$$



May '72

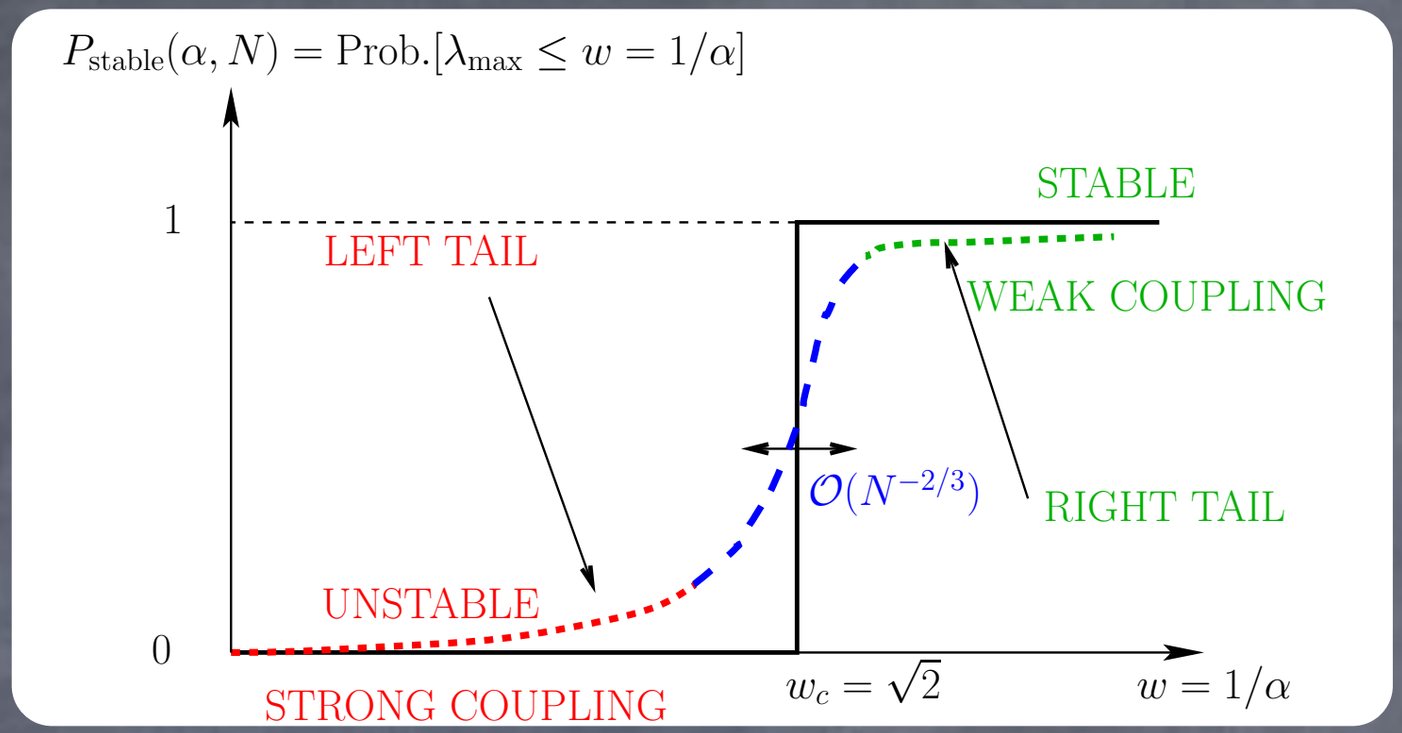
Stable/Unstable transition for large systems

- What happens for finite but large systems, of size $N \gg 1$?



- ✓ Is there any thermodynamic sense to this transition ?
- ✓ What is the analogue of the free energy ?
- ✓ What is the order of this transition ?

For large but finite N : summary of the results



$$\Pr.[\lambda_{\max} < w] = F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right] & , w < \sqrt{2} \;\&\; |w - \sqrt{2}| \sim \mathcal{O}(1) \\ \mathcal{F}_\beta\left(\sqrt{2} N^{\frac{2}{3}} (w - \sqrt{2})\right) & , \; |w - \sqrt{2}| \sim \mathcal{O}(N^{-\frac{2}{3}}) \\ 1 - \exp\left[-\beta N \Phi_+(w)\right] & , w > \sqrt{2} \;\&\; |w - \sqrt{2}| \sim \mathcal{O}(1) \end{cases}$$

and $\beta = 1$ (GOE) for May's model

see also Mylène 's and Gaëtan 's talks

- ✓ Crossover function: $\mathcal{F}_\beta(x)$ Tracy & Widom '94
- ✓ Large tail rate functions: $\Phi_\pm(w)$ Ben-Arous, Dembo & Guionnet '01 , Dean & Majumdar '06, Majumdar & Vergassola '09

Exact large deviation functions

- Using Coulomb gas techniques and saddle point calculation for $N \gg 1$

- Left deviation function $\Phi_-(w) = \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6}$
$+ 27\left(\ln 18 - 2\ln\left(w + \sqrt{w^2+6}\right)\right)\Big]\,,\ w < \sqrt{2}$

when $w \to \sqrt{2}^-$ $\quad \Phi_-(w) \sim \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3\,,\ w \to \sqrt{2}^-$

Dean & Majumdar '06

- Right deviation function $\Phi_+(w) = \frac{1}{2}w\sqrt{w^2-2} + \ln\left[\frac{w - \sqrt{w^2-2}}{\sqrt{2}}\right]$

when $w \to \sqrt{2}^+$ $\quad \Phi_+(w) \sim \frac{2^{7/4}}{3}(w - \sqrt{2})^{3/2}\,,\ w \to \sqrt{2}^+$

Ben-Arous, Dembo & Guionnet '01 , Majumdar & Vergassola '09

Third order phase transition

$$\Pr.[\lambda_{\max} < w] = F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right] & , w < \sqrt{2}\,, \quad \text{unstable} \\ 1 - \exp\left[-\beta N \Phi_+(w)\right] & , w > \sqrt{2}\,, \quad \text{stable} \end{cases}$$

- This implies the behavior of the free energy

$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} \Phi_-(w) \propto (\sqrt{2} - w)^3\,, & w < \sqrt{2}\,, \\ 0 & w > \sqrt{2} \end{cases}$$

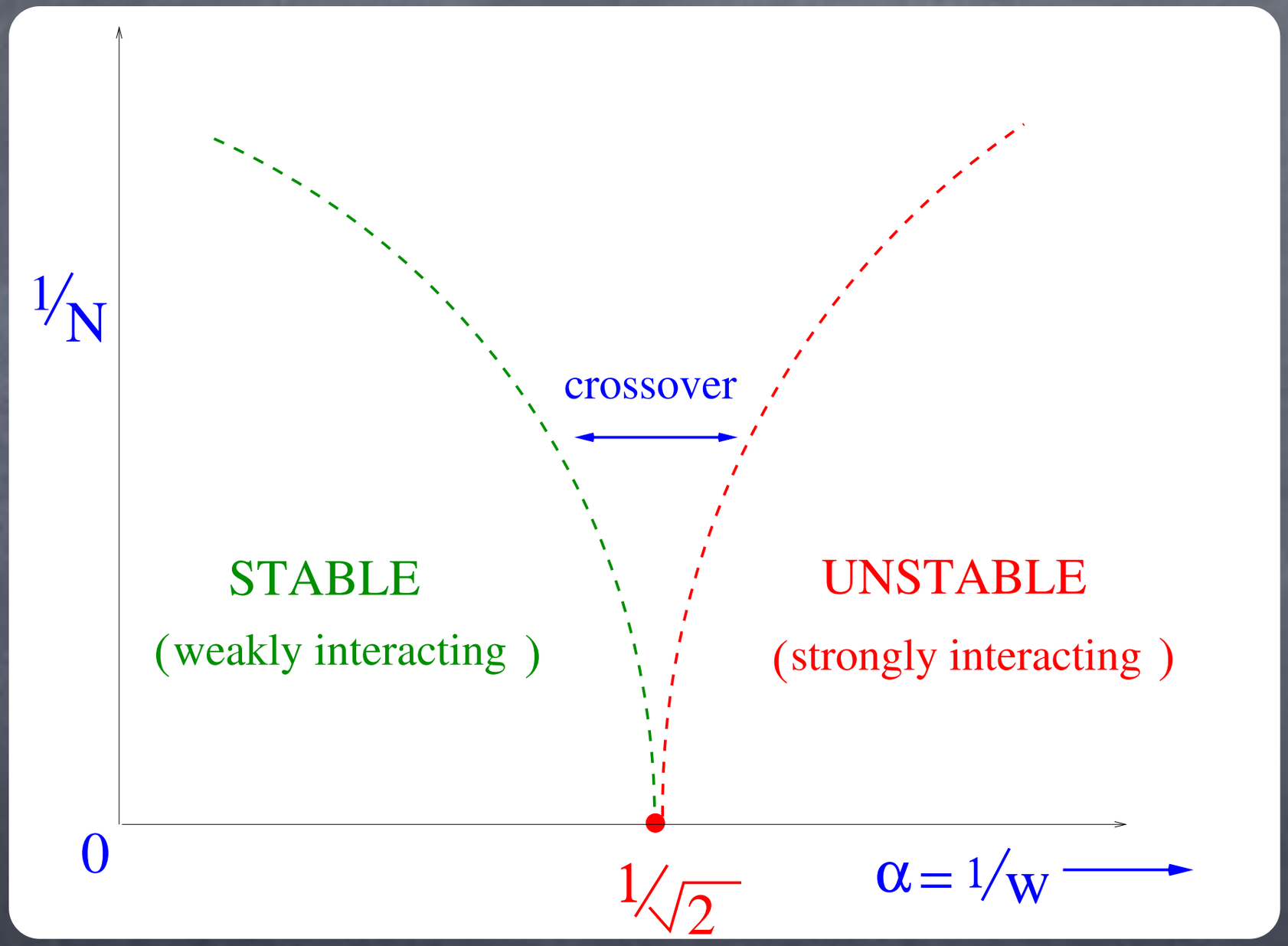
- The third derivative of the free energy is discontinuous

$\Longrightarrow$ Third order phase transition

- The crossover, for finite N , between the two phases is described by the Tracy-Widom distribution

S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014)

Third order phase transition



S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014)

Similar third order phase transition in gauge theory

PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

Possible third-order phase transition in the large-N lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 10 July 1979)

The large-N limit of the two-dimensional $U(N)$ (Wilson) lattice gauge theory is explicitly evaluated for all fixed $\lambda = g^2N$ by steepest-descent methods. The λ dependence is discussed and a third-order phase transition, at $\lambda = 2$, is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large-N four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

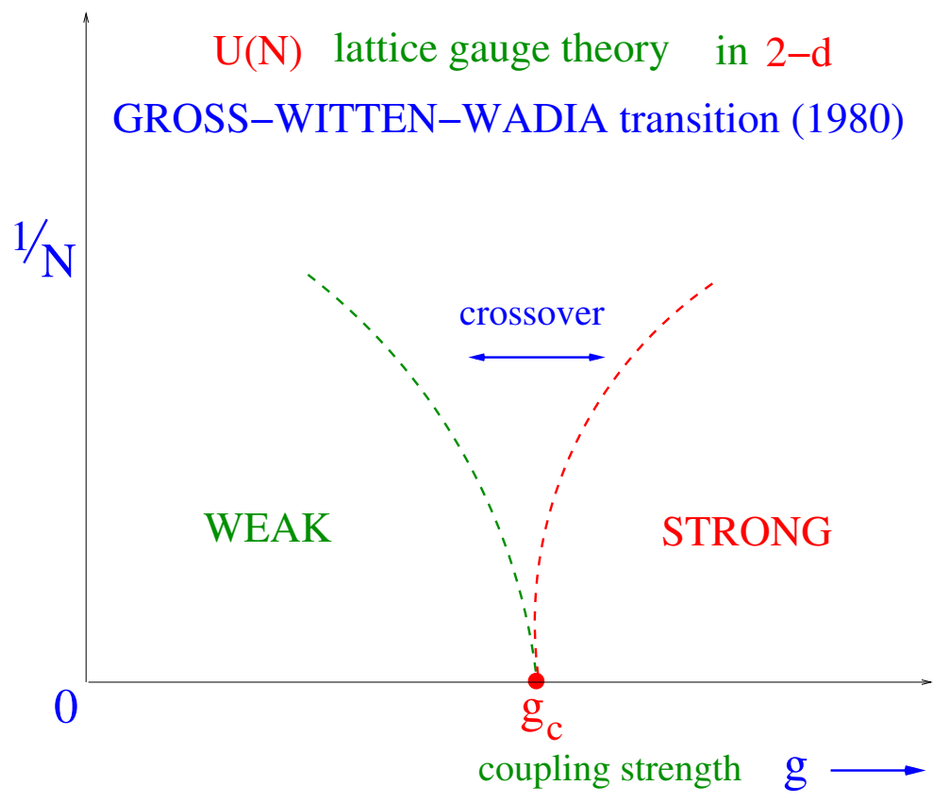
Volume 93B, number 4 PHYSICS LETTERS 30 June 1980

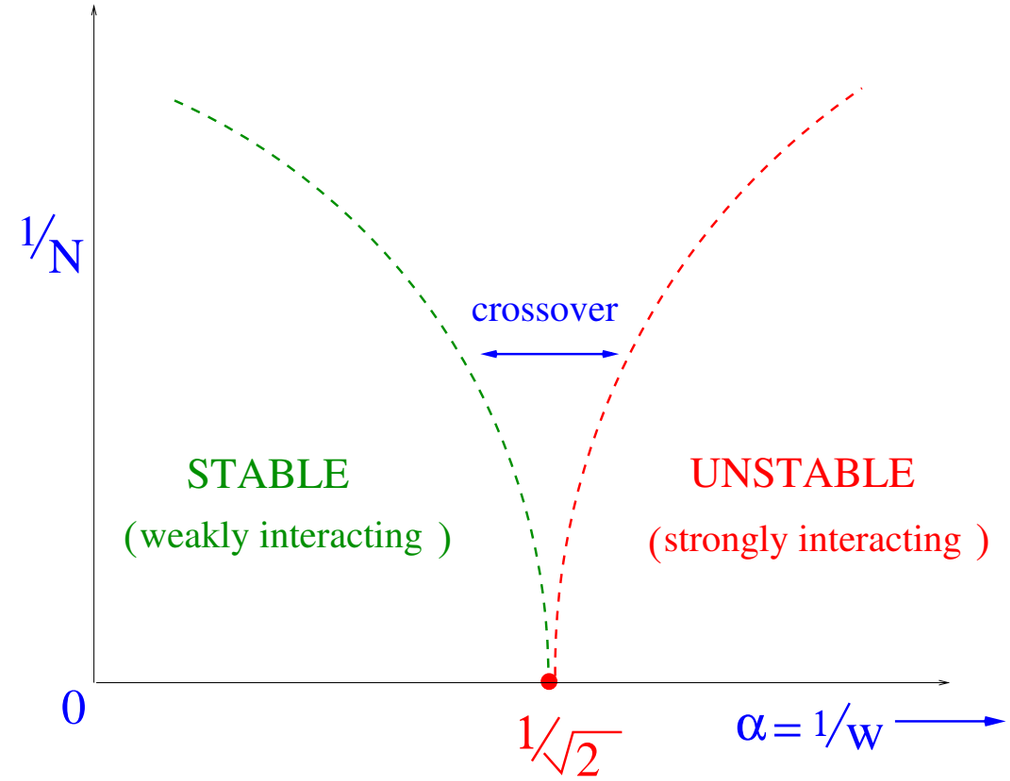
$N = \infty$ PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE MODEL LATTICE GAUGE THEORIES *

Spenta R. WADIA
The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Received 27 March 1980

Similar third order phase transition in gauge theory





Similar transition in $U(N)$ lattice gauge theory

Unstable phase = strong coupling phase of Yang-Mills gauge theory
Stable phase = weak coupling phase of Yang-Mills gauge theory

Tracy-Widom ditribution describes the crossover between the two regimes (at finite but large N): double scaling regime

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- Largest eigenvalue $\lambda_{\max}$ of a Gaussian random matrix
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- Proba. distrib. func. (PDF) of $\lambda_{\max}$: Coulomb gas (CG) (with a wall)
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  - ⟹ Third order phase transition
- Similar third order transition in other systems
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Gaussian random matrix models

- $N \times N$ random matrix : $\mathbf{M} \equiv M_{i,j}$

- Standard Dyson 's ensembles: Orthogonal, Unitary, Symplectic
  (GOE) (GUE) (GSE)

- Gaussian probability measure $P(\mathbf{M})d\mathbf{M} \propto \exp\left[-\frac{\beta}{2}N\mathrm{Tr}(\mathbf{M}^\dagger\mathbf{M})\right]d\mathbf{M}$

- Joint PDF of the real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ Wigner '51

$$P_{\text{joint}}(\lambda_1, \lambda_2, \cdots, \lambda_N) = \frac{1}{Z_N}\exp\left[-\frac{\beta}{2}N\sum_{i=1}^{N}\lambda_i^2\right]\prod_{i<j}|\lambda_i - \lambda_j|^\beta$$

with $\beta = 1$(GOE), $\beta = 2$(GUE) and $\beta = 4$(GSE)

- Partition function

$$Z_N = \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \exp\left[-\frac{\beta}{2}N\sum_{i=1}^{N}\lambda_i^2\right]\prod_{i<j}|\lambda_i - \lambda_j|^\beta$$

Coulomb Gas picture and Wigner semi-circle

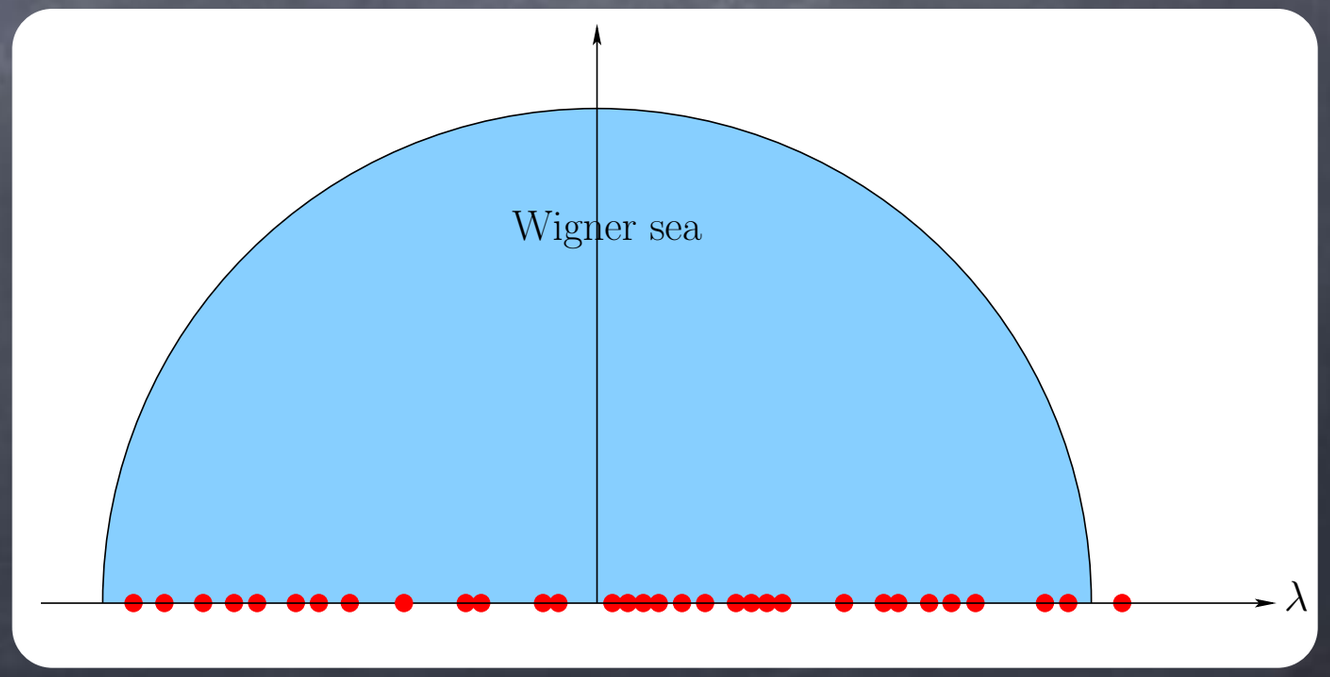
- Rewrite the partition function as

$$Z_N(\beta) = \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \exp\left[-\beta\left(\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2}\sum_{i\neq j}\ln|\lambda_i - \lambda_j|\right)\right]$$

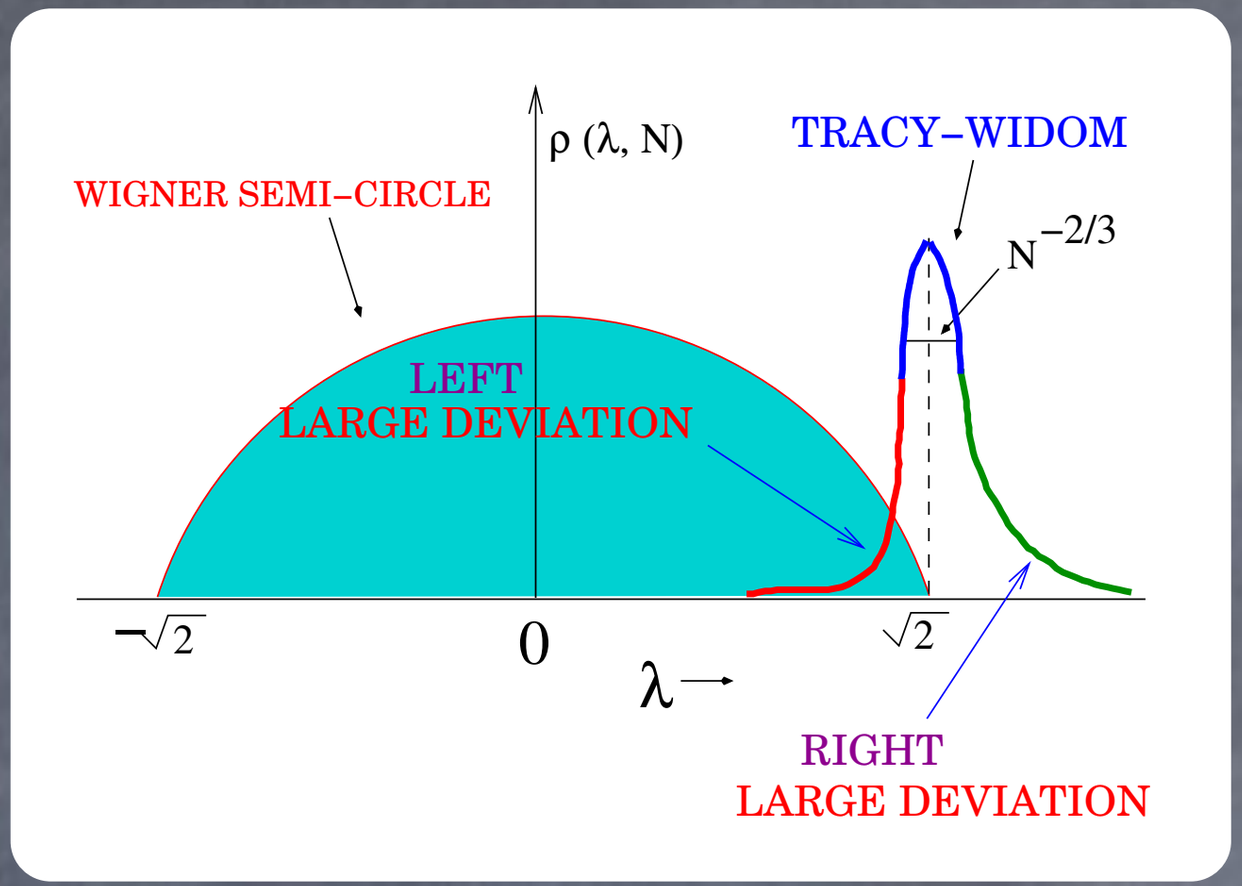
- 2-d Coulomb gas confined to a line, with β the inverse temperature
- Typical scale of the eigenvalues: $N^2\lambda_{\rm typ}^2 \sim N^2 \Longrightarrow \lambda_{\rm typ} \sim \mathcal{O}(1)$
- Mean density of eigenvalues

$$\rho_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(\lambda - \lambda_i)\rangle$$

$$\xrightarrow[N\to\infty]{} \rho_{\rm SC}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$



Tracy-Widom distributions for $\lambda_{\max}$



- In the limit $N \to \infty$, $\lambda_{\max} \to \sqrt{2}$
- The typical scale of fluctuations of $\lambda_{\max}$ can be obtained via

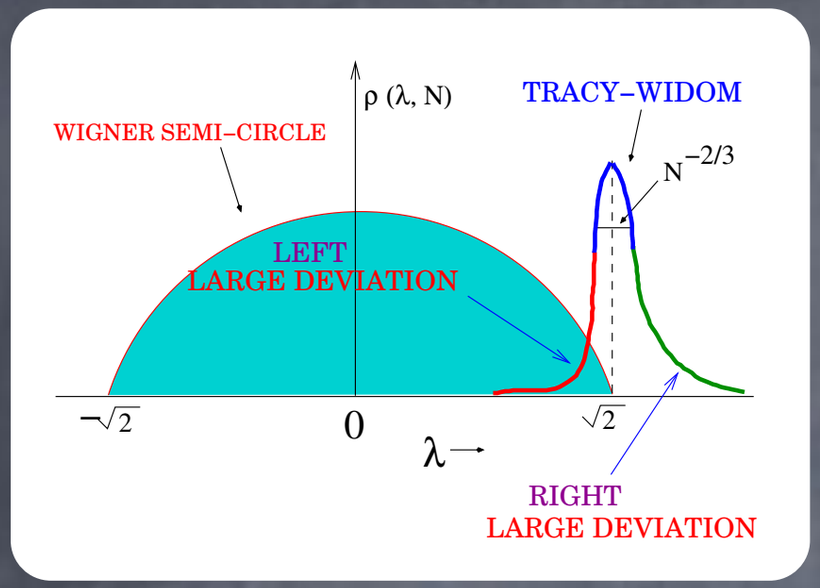
$$\int_{\lambda_{\max}}^{\sqrt{2}} \rho_{\rm SC}(\lambda)\, d\lambda \sim 1/N \quad \Longrightarrow \quad |\lambda_{\max} - \sqrt{2}| \sim \mathcal{O}(N^{-2/3})$$

- Tracy-Widom distributions

$$\Pr.[\lambda_{\max} \leq w] \longrightarrow \mathcal{F}_\beta(\sqrt{2}(w - \sqrt{2})N^{2/3})$$

Tracy & Widom '94

Tracy-Widom distributions for $\lambda_{\max}$



Asymmetric behavior of the PDF

$$\mathcal{F}'_\beta(x) \approx \begin{cases} \exp\left[-\dfrac{\beta}{24}|x|^3\right], & x \to -\infty \\[2ex] \exp\left[-\dfrac{2\beta}{3}x^{3/2}\right], & x \to +\infty\,, \end{cases}$$

- Explicit expression in terms of a Painlevé II transcendent

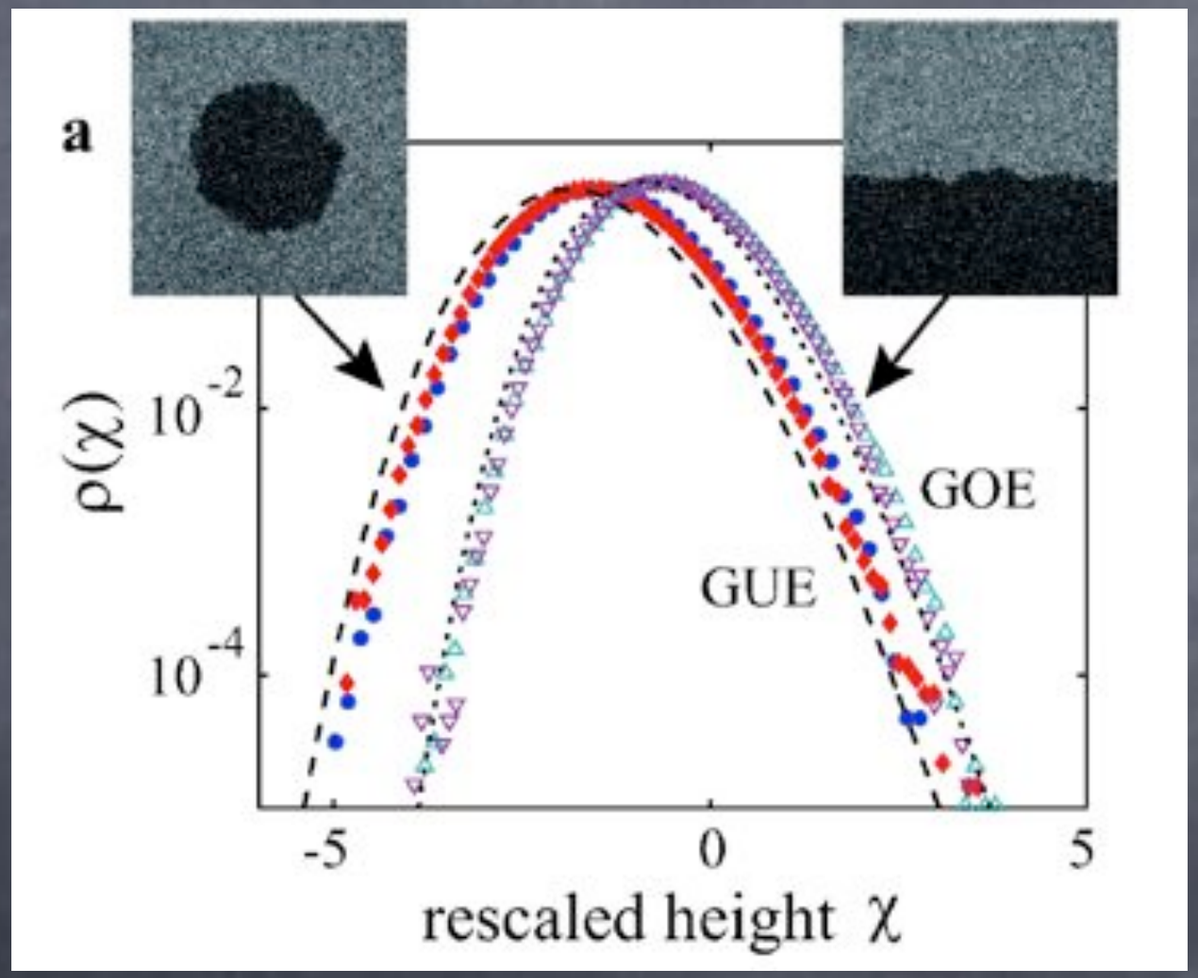
Tracy & Widom '94 $\qquad q''(x) = xq(x) + 2q^3(x)\,,\ q(x) \underset{x\to\infty}{\sim} \mathrm{Ai(x)}$

- Ubiquity of Tracy-Widom distributions
  - ✓ largest eigenvalue of Wishart-Laguerre matrices
  - ✓ longest increasing subsequence of random permutations
  - ✓ directed polymers and growth models in the KPZ universality class
  - ✓ continuum KPZ equation
  - ✓ sequence alignment problems
  - ✓ mesoscopic fluctuations in quantum dots
  - ✓ maximal height of non-intersecting Brownian motions

Baik, Borodin, Calabrese, Comtet, Corwin, Deift, Dotsenko, Edelman, Ferrari, Forrester, Johansson, Johnstone, Le Doussal, Majumdar, Nadal, Nechaev, O'Connell, Péché, Prähofer, Quastel, Rains, Rosso, Sano, Sasamoto, Spohn, Takeuchi, Virág

Ubiquity of Tracy-Widom distributions

- Experimental observation of TW distributions for GOE $(\beta = 1)$ and GUE $(\beta = 2)$ in liquid crystals experiments

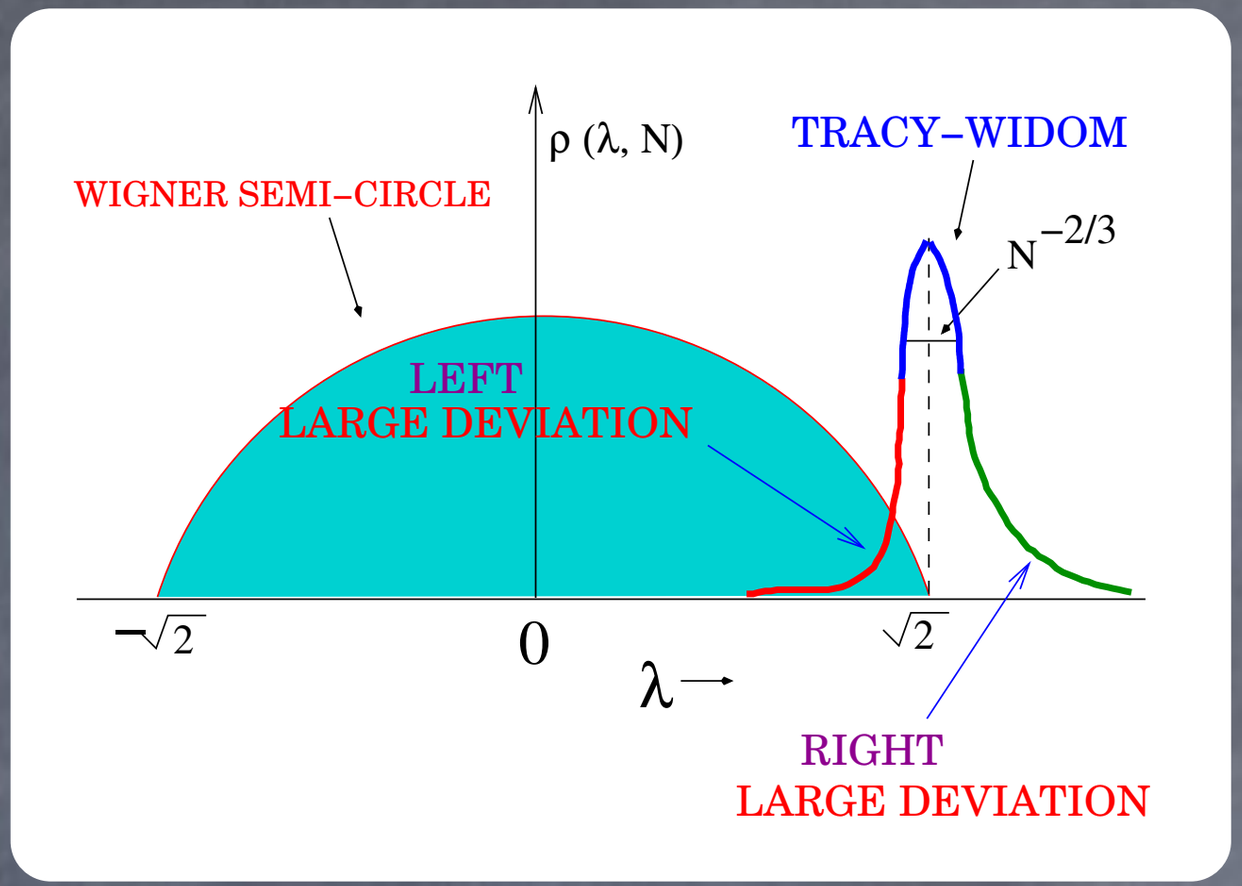


Takeuchi & Sano '10

Takeuchi, Sano, Sasamoto & Spohn '11

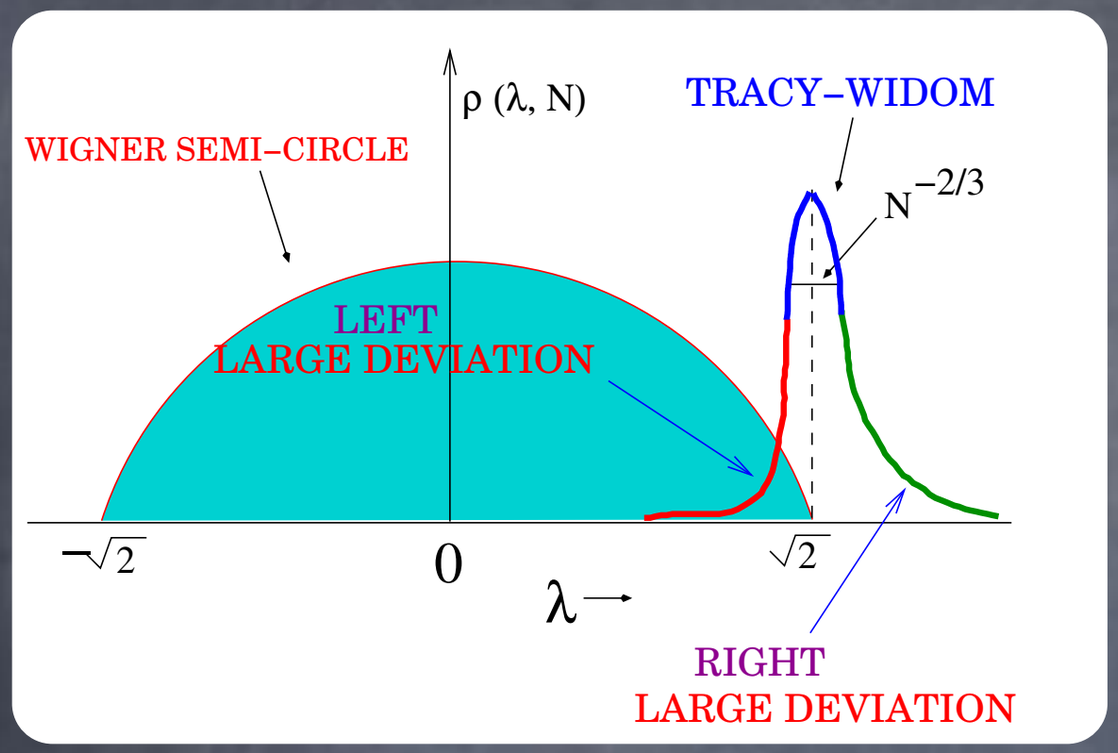
from Takeuchi, Sano, Sasamoto & Spohn, Sci. Rep. (Nature) 1, 34 (2011)

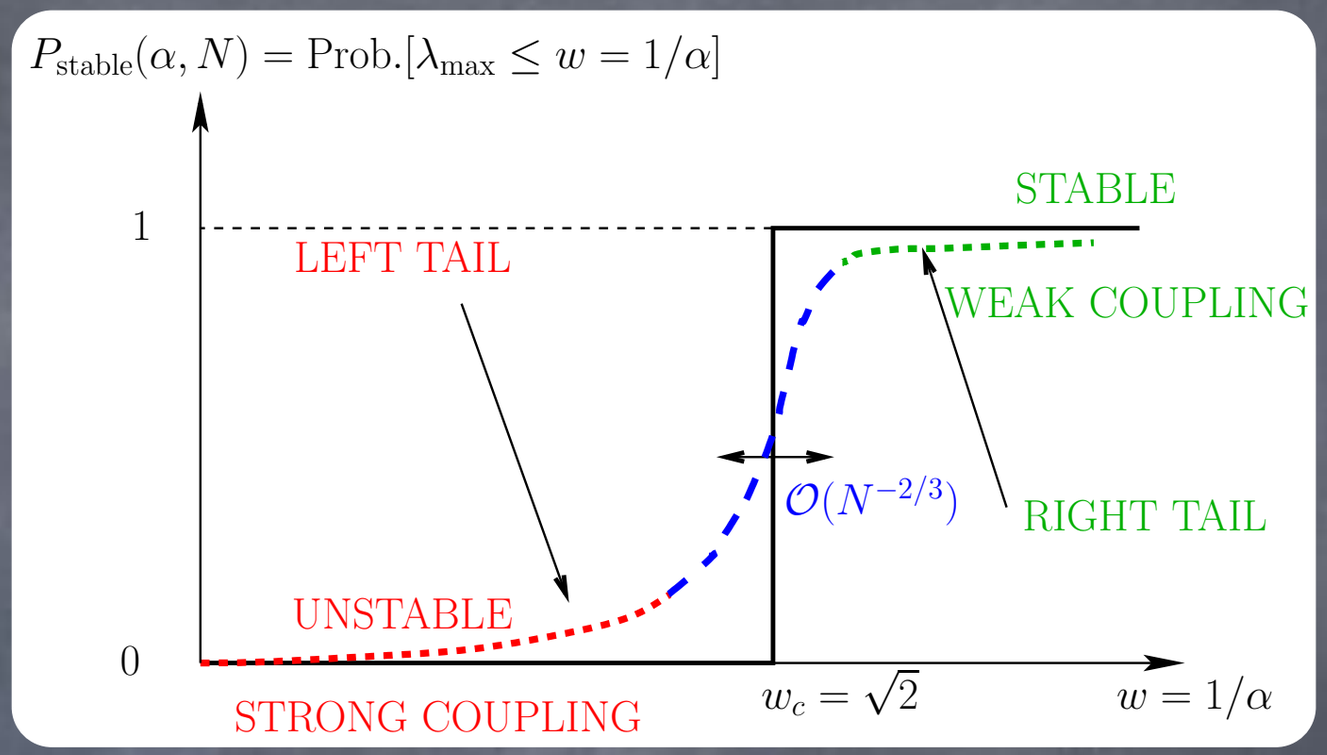
Beyond TW distributions: large deviations of $\lambda_{\max}$



- TW-distributions describe the typical fluctuations, $|\lambda_{\max} - \sqrt{2}| \sim \mathcal{O}(N^{-2/3})$
- What about the $\left\{\begin{array}{l}\text{atypical fluctuations, } |\lambda_{\max} - \sqrt{2}| \sim \mathcal{O}(1)\text{?} \\ \text{large deviations}\end{array}\right.$

Beyond TW distributions: large deviations of $\lambda_{\max}$





$$\Pr.[\lambda_{\max} < w] = F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right] & , w < \sqrt{2} \;\&\; |w - \sqrt{2}| \sim \mathcal{O}(1) \\ \mathcal{F}_\beta\left(\sqrt{2} N^{\frac{2}{3}} (w - \sqrt{2})\right) & , \qquad |w - \sqrt{2}| \sim \mathcal{O}(N^{-\frac{2}{3}}) \\ 1 - \exp\left[-\beta N \Phi_+(w)\right] & , w > \sqrt{2} \;\&\; |w - \sqrt{2}| \sim \mathcal{O}(1) \end{cases}$$

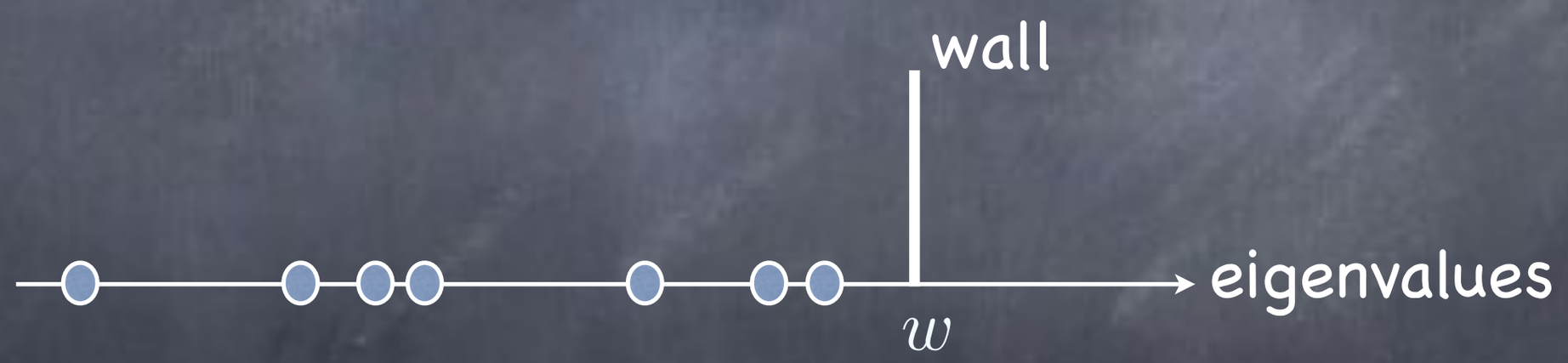
Coulomb Gas with a wall

Cumulative distribution function of $\lambda_{\max}$

$$\Pr.[\lambda_{\max} \leq w] = \Pr.[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)}$$

$$\boxed{Z_N(w) = \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp\left[-\beta N^2 E[\{\lambda_i\}]\right]}$$

$$E[\{\lambda_i\}] = \frac{1}{2N}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2N^2}\sum_{j\neq k}\log|\lambda_j - \lambda_k|$$



What happens when the wall is moved ?

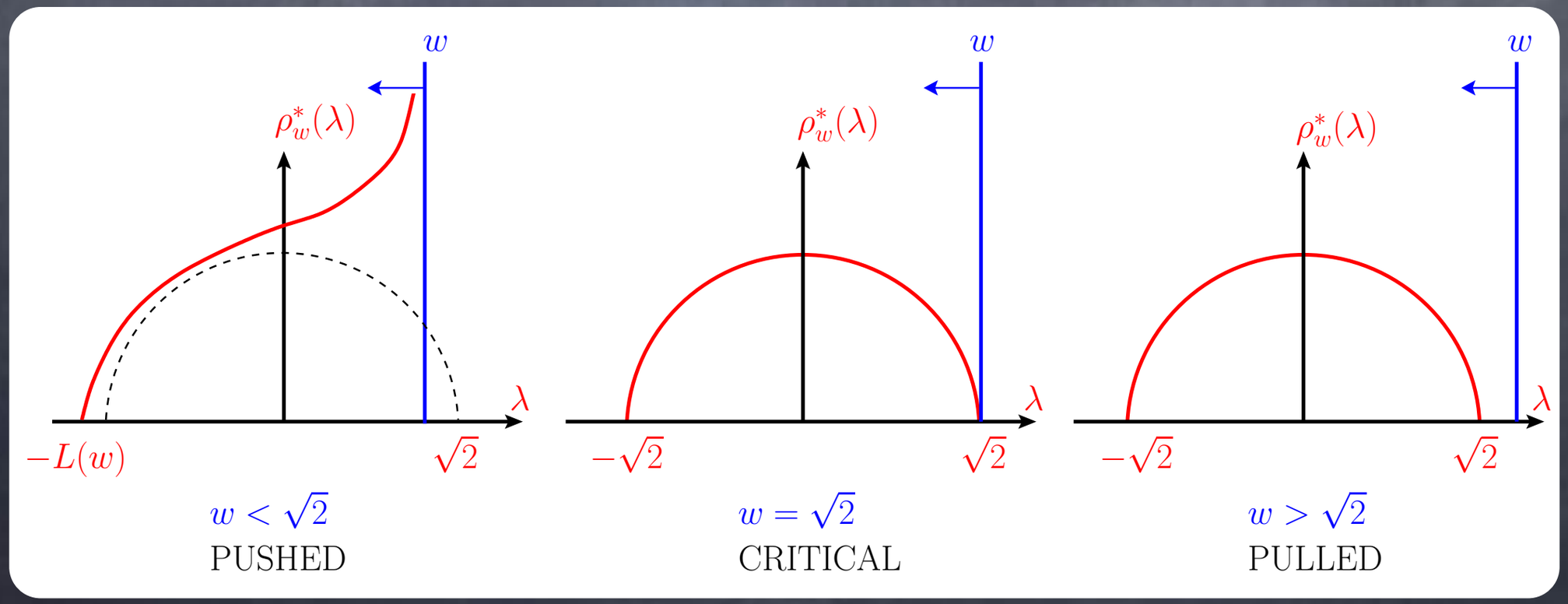
Pushed vs. pulled Coulomb gas

$$Z_N(w) = \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp\left[-\beta N^2 E[\{\lambda_i\}]\right]$$

$$E[\{\lambda_i\}] = \frac{1}{2N}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2N^2}\sum_{j\neq k}\log|\lambda_j - \lambda_k|$$

- Saddle point analysis Dean & Majumdar '06, '08

Mean density of eigenvalues in presence of the wall $\rho_w^*(\lambda)$



Left large deviation function

$$F(w,N) = \Pr.[\lambda_{\max} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim \exp[-\beta N^2 \Phi_-(w)] \ , \ w < \sqrt{2}$$

i.e. $$\lim_{N\to\infty} -\frac{1}{\beta N^2} \ln F(w,N) = \Phi_-(w)$$

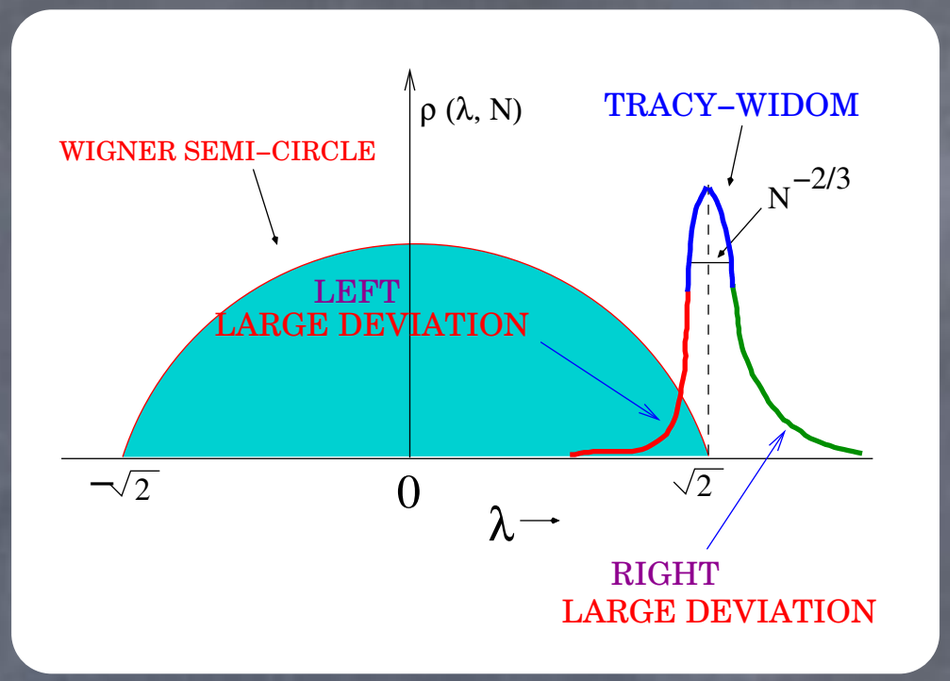
- Physically, $N^2\Phi_-(w)$ is the energy to push the Coulomb gas

- Left deviation function $$\Phi_-(w) = \frac{1}{108}\Big[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln 18 - 2\ln\left(w + \sqrt{w^2+6}\right)\right)\Big] \ , \ w < \sqrt{2}$$

when $w \to \sqrt{2}^-$ $$\Phi_-(w) \sim \frac{1}{6\sqrt{2}}(\sqrt{2} - w)^3 \ , \ w \to \sqrt{2}^-$$

Dean & Majumdar '06, '08

Matching between large deviation & and Tracy-Widom



when $w \to \sqrt{2}^-$ $\quad \Phi_-(w) \sim \frac{1}{6\sqrt{2}}(\sqrt{2}-w)^3 \ , \ w \to \sqrt{2}^-$

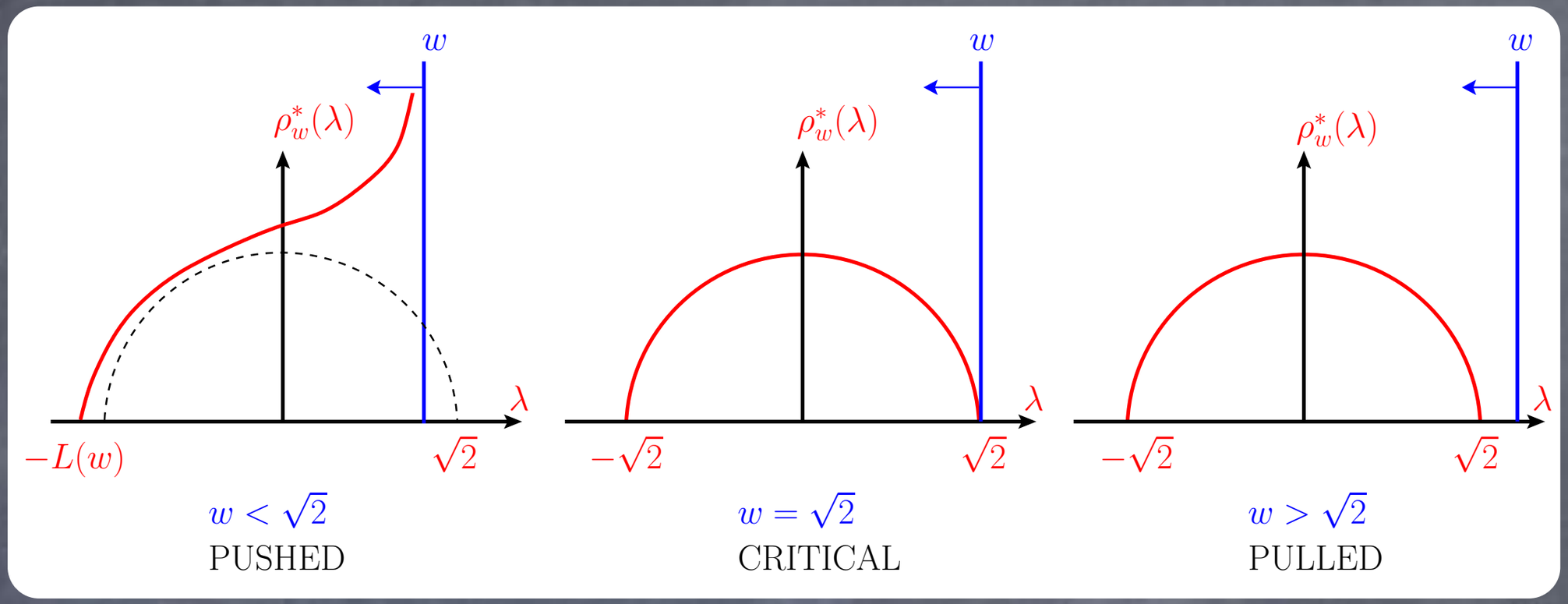
⟹ matches with the left tail of the Tracy-Widom distribution

$$\Pr.[\lambda_{\max} \leq w] \sim \exp[-\beta N^2 \Phi_-(w) \sim] \sim \exp\left[-\frac{\beta}{24}\left[2^{1/2}N^{2/3}(\sqrt{2}-w)\right]^3\right]$$

which coincides with the left tail of the TW distribution

$$\mathcal{F}_\beta(x) \sim \exp\left(-\frac{\beta}{24}|x|^3\right) \ , \ x \to -\infty$$

Right large deviation function



- The saddle point equation yields a trivial result for $w > \sqrt{2}$

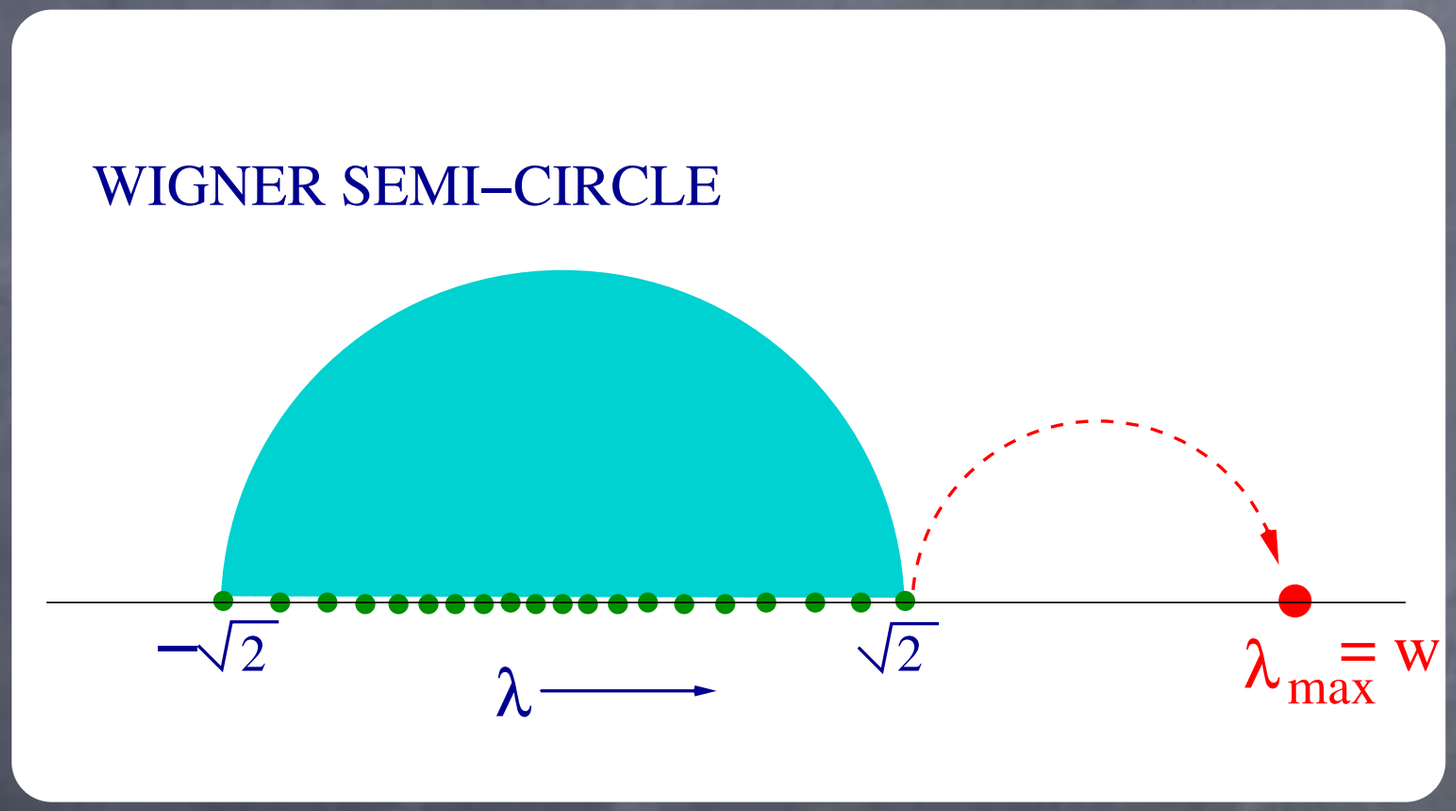
$$F(w, N) = \Pr.[\lambda_{\max} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim 1 \ , \ w > \sqrt{2}$$

$\Longrightarrow$ Non trivial corrections requires a different approach

Right tail: pulled Coulomb gas

$$F(w,N) = \Pr.[\lambda_{\max} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim 1 - \exp(-N\Phi_+(w)) \ , \ w > \sqrt{2}$$

$N\Phi_+(w)$: energy to pull a single charge out of the Wigner sea



$$\Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[\frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right]$$

Ben-Arous, Dembo & Guionnet '01,
Majumdar & Vergassola '09

when $w \to \sqrt{2}^+$ $\quad \Phi_+(w) \sim \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2} \ , \ w \to \sqrt{2}^+$

$\Longrightarrow$ matches with the right tail of the Tracy-Widom distribution

Third order phase transition

$$\Pr.[\lambda_{\max} < w] = F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right] & , w < \sqrt{2}\,, \quad \text{unstable} \\ 1 - \exp\left[-\beta N \Phi_+(w)\right] & , w > \sqrt{2}\,, \quad \text{stable} \end{cases}$$

- This implies the behavior of the free energy

$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} \Phi_-(w) \propto (\sqrt{2} - w)^3\,, & w < \sqrt{2}\,, \\ 0 & w > \sqrt{2} \end{cases}$$

- The third derivative of the free energy is discontinuous

$\Longrightarrow$ Third order phase transition

- The crossover, for finite N, between the two phases is described by the Tracy-Widom distribution

S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014)

Laguerre-Wishart random matrices

- Let X be a $M \times N$ Gaussian random matrix (real or complex)

  with $M \geq N$

- $W = X^\dagger X$ is a $N \times N$ symmetric or Hermitian Gaussian matrix

- Joint probability density function of the N real eigenvalues $\lambda_i' s$

$$P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i=1}^{N} \lambda_i^a \prod_{i<j} |\lambda_i - \lambda_j|^\beta \mathrm{e}^{-N\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \ , \ \lambda_i \geq 0 \ , \ \forall i$$

where $\beta = 1$ (real) and $a = \frac{\beta}{2}(M - N + 1) - 1$

$\beta = 2$ (complex)

- Largest eigenvalue in the Wishart-Laguerre ensemble

$$\lambda_{\max} = \max_{1 \leq i \leq N} \lambda_i$$

Largest eigenvalue of Laguerre-Wishart random matrices

$$P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i=1}^{N} \lambda_i^a \prod_{i<j} |\lambda_i - \lambda_j|^\beta \mathrm{e}^{-N\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \ , \ \lambda_i \geq 0 \ , \ \forall i$$

- Density of eigenvalues: Marčenko-Pastur law

$$\frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i) \underset{N\to\infty}{\longrightarrow} \rho_{\mathrm{MP}}(\lambda) = \frac{1}{2\pi\lambda}\sqrt{(\lambda - \lambda_-)(\lambda_+ - \lambda)}$$

- Large deviations of $\lambda_{\max} = \max_{1\leq i\leq N} \lambda_i$:

$$\Pr.[\lambda_{\max} < w] = F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Psi_-(w)\right] & , w < \lambda_+ \ \& \ |w - \lambda_+| \sim \mathcal{O}(1) \\ \mathcal{F}_\beta\left(a_\beta N^{\frac{2}{3}}(w - \lambda_+)\right) & , \ |w - \lambda_+| \sim \mathcal{O}(N^{-\frac{2}{3}}) \\ 1 - \exp\left[-\beta N \Psi_+(w)\right] & , w > \lambda_+ \ \& \ |w - \lambda_+| \sim \mathcal{O}(1) \end{cases}$$

with $\Psi_-(w) \underset{w\to\lambda_+}{\sim} (\lambda_+ - w)^3$

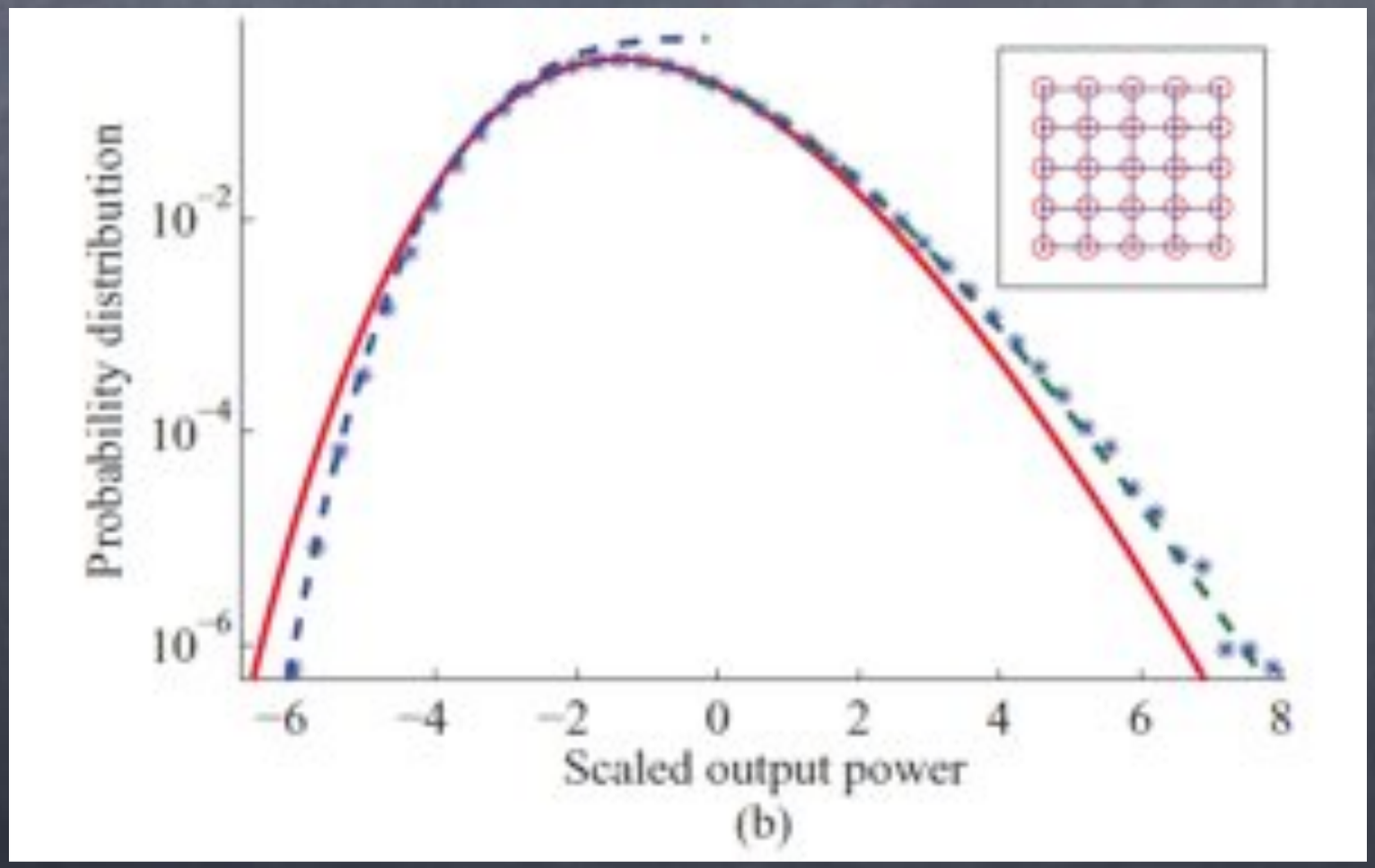
Majumdar & Vergassola '09
Vivo, Majumdar & Bohigas '07

Largest eigenvalue of Laguerre-Wishart random matrices in experiments

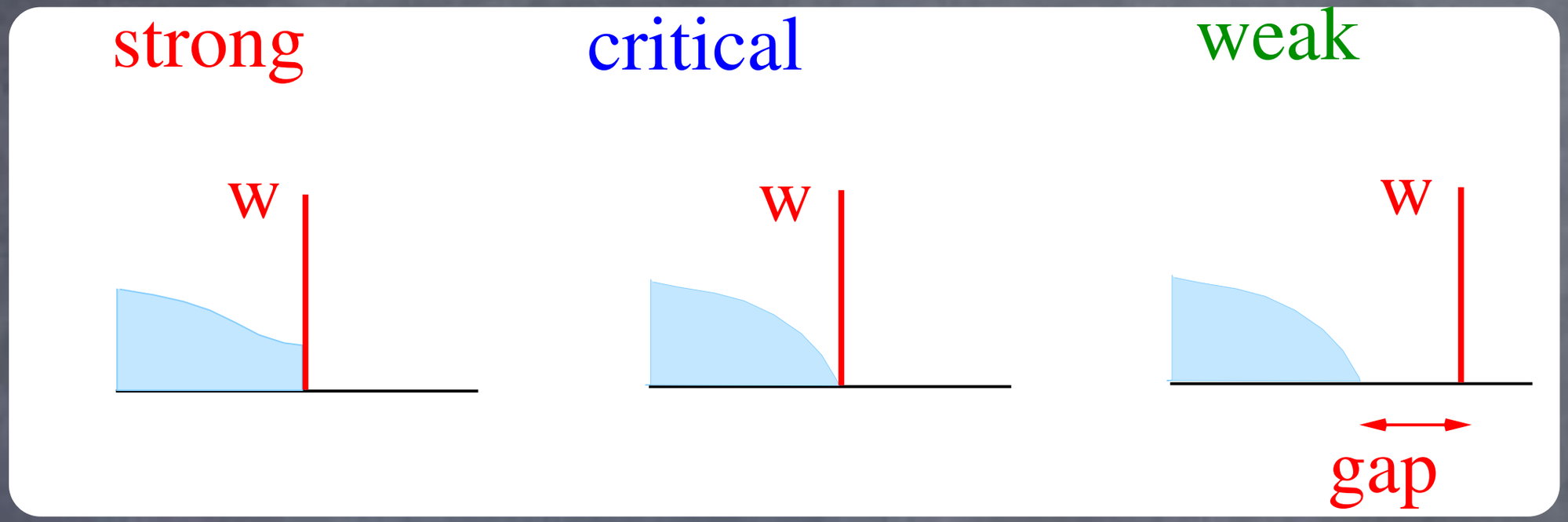
PHYSICAL REVIEW E 85, 020101(R) (2012)

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson*
Weizmann Institute of Science, Department of Physics of Complex Systems, Rehovot 76100, Israel
(Received 16 December 2011; published 1 February 2012)



Third order phase transition: basic mechanism



A third order phase transition between a weak coupling and a strong coupling phase occurs when the gap, between the soft edge of a Coulomb droplet with density vanishing as a square-root at the edge and a hard wall at w, vanishes as one tunes a control parameter through its critical value.

S. N. Majumdar, G. S., J. Stat. Mech. P01012 (2014)

Outline

- Largest eigenvalue $\lambda_{\max}$ of a Gaussian random matrix

  ✓ Application to the stability of large complex system

  ⟹ Hint at a phase transition

- Proba. distrib. func. (PDF) of $\lambda_{\max}$: Coulomb gas (CG) (with a wall)

  ✓ Tracy-Widom distribution

  ✓ Physics of large deviation tails

  ⟹ Third order phase transition

- Similar third order transition in other systems

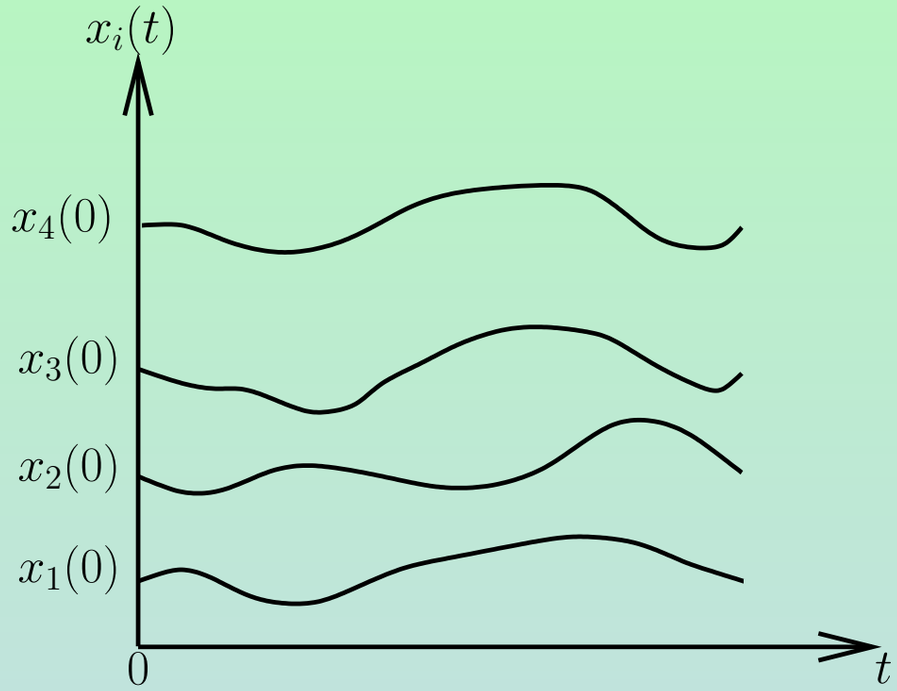
  ✓ Non-intersecting Brownian motions

- Summary and conclusion

Non-intersecting Brownian motions

- N non intersecting Brownian motions in one-dimension

$$x_1(t) < \quad x_2(t) \quad < \ldots < x_N(t) \ ,$$
$$\forall t \quad \geq 0$$



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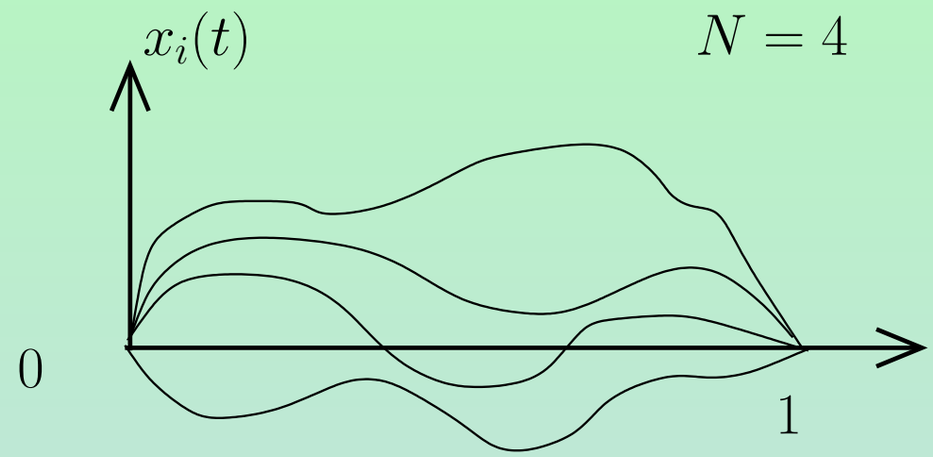

watermelons

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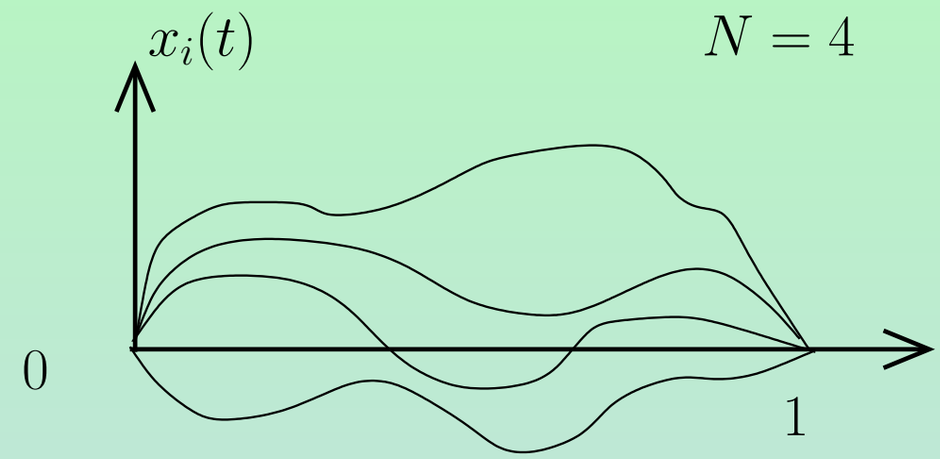
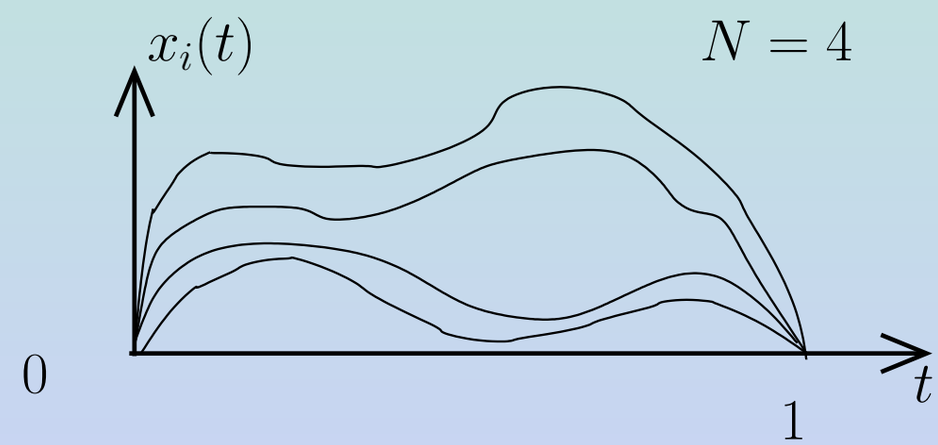


watermelons



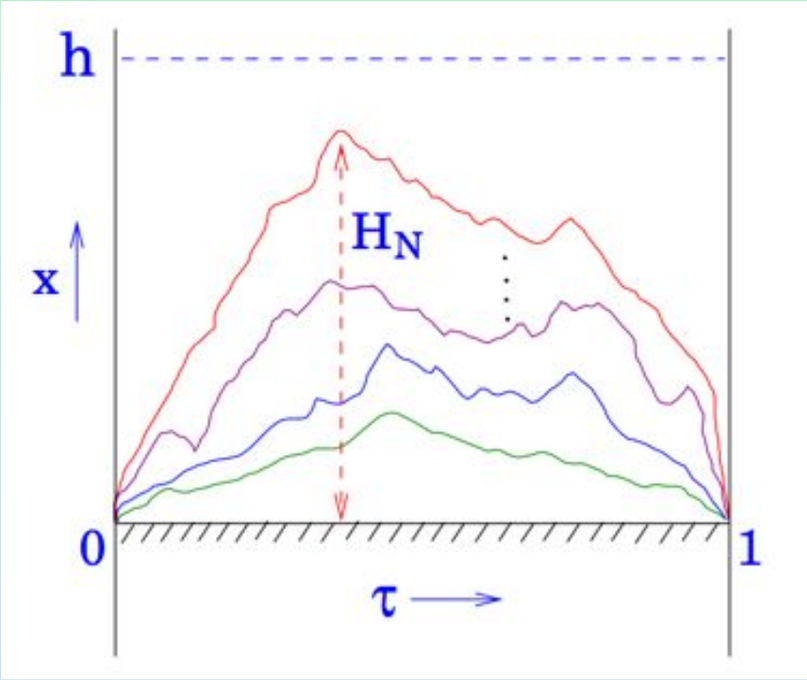
watermelons "with a wall"

Tracy & Widom '07

An exact expression for the PDF of the maximal height

- Distribution of the maximal height $\mathcal{H}_N$

$$F_N(h) \quad = \quad \mathbb{P}\left[x_N(\tau) \leq h\,,\ \forall\, 0 \leq \tau \leq 1\right]$$



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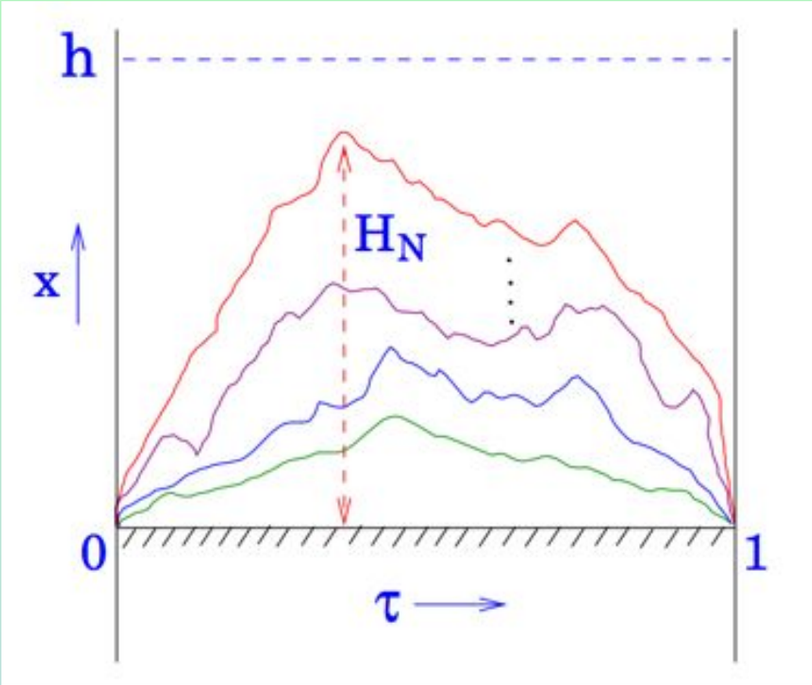

- Exact result for finite N G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(h) \quad = \quad \frac{A_N}{h^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^{N} n_i^2}$$

$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2+\frac{N}{2}} \prod_{j=0}^{N-1} \Gamma(2+j)\Gamma(\frac{3}{2}+j)}$$

see also T. Feierl, M. Katori *et al.* '08

Large N asymptotics

- $F_N(h) \equiv$ partition function of a $2d$ gauge theory with symmetry group $\mathrm{Sp}(2N)$, similar to the one studied by Douglas & Kazakov

P. J. Forrester, S. N. Majumdar, G. Schehr '11

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  P. J. Forrester, S. N. Majumdar, G. Schehr '11

- Saddle point analysis for large N, $h = \tilde{h}\sqrt{N}$

$$F_N(h) \propto \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2\tilde{h}^2 N}\sum_{i=1}^{N} n_i^2}$$

when $N \to \infty$, $\dfrac{n_i}{2N} := x_i$ are continuous variables

$$F_N(\tilde{h}\sqrt{2N}) \sim \int \mathcal{D}\tilde{\rho}(x) e^{-N^2 S[\tilde{\rho}]} \ , \ \tilde{\rho}(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i) \leq 2$$

$$S[\tilde{\rho}] = \frac{\pi^2}{\tilde{h}^2}\int_0^a dx\, x^2 \tilde{\rho}(x) - \int_0^a dx \int_0^a dx'\, \tilde{\rho}(x)\tilde{\rho}(x') \ln|x^2 - x'^2| + C\left[\int_0^a \tilde{\rho}(x)\, dx - 1\right]$$

Large N asymptotics: saddle point analysis

- Constrained saddle point

$$\int \mathcal{D}\tilde{\rho}(x) e^{-N^2 S[\tilde{\rho}]} \sim \exp\left(-N^2 S[\rho^*]\right) \ , \ \left.\frac{\delta S[\rho]}{\delta \tilde{\rho}(x)}\right|_{\tilde{\rho}=\rho^*} = 0$$

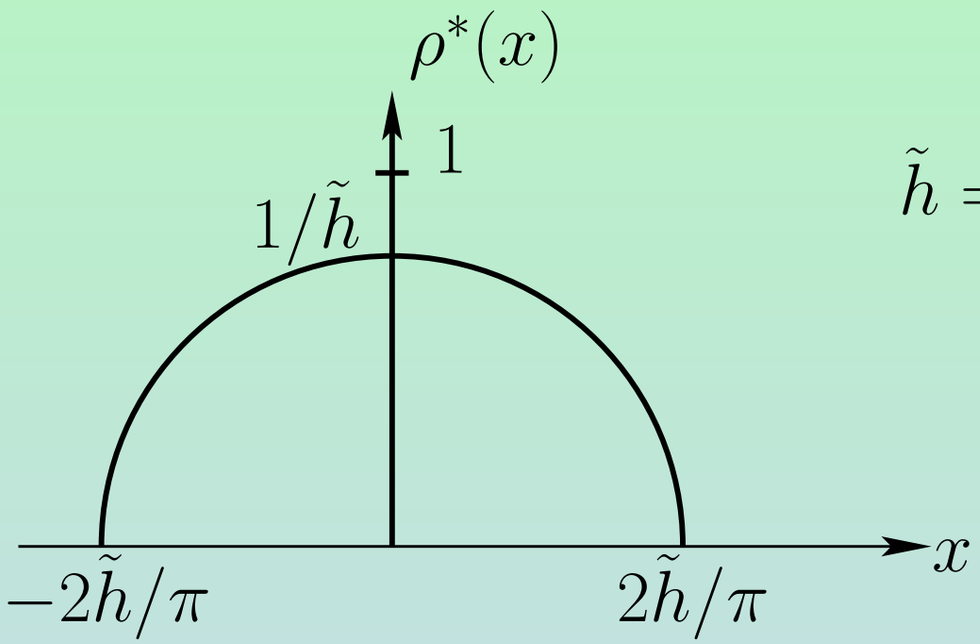
$\Rightarrow$ integral equation for $\rho^*(x)$

$$\frac{\pi^2}{2h^2} x^2 - 2 \int_{-a}^{a} \rho^*(x') \ln|x - x'| \, dx' + C' = 0$$

$$\int_{-a}^{a} \rho^*(x) dx = 1 \ , \ \rho^*(x) \leq 1 \ , \forall x \in [-a, a]$$

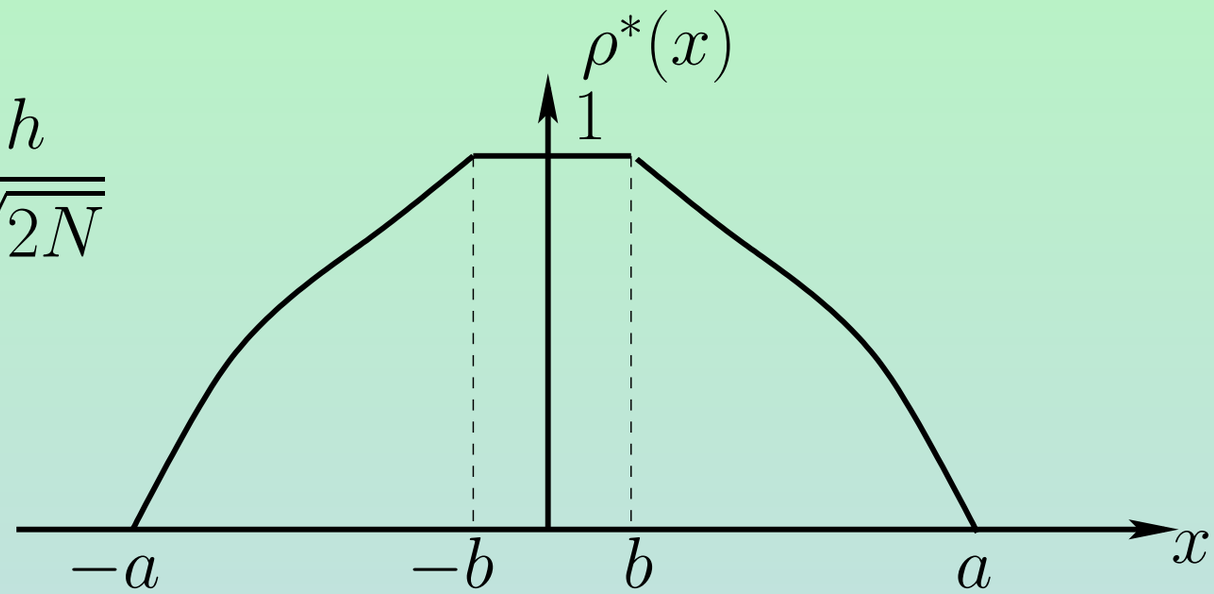
Large N asymptotics: saddle point analysis

- Douglas-Kazakov transition



$\tilde{h} > 1$

$$\tilde{h} = \frac{h}{\sqrt{2N}}$$



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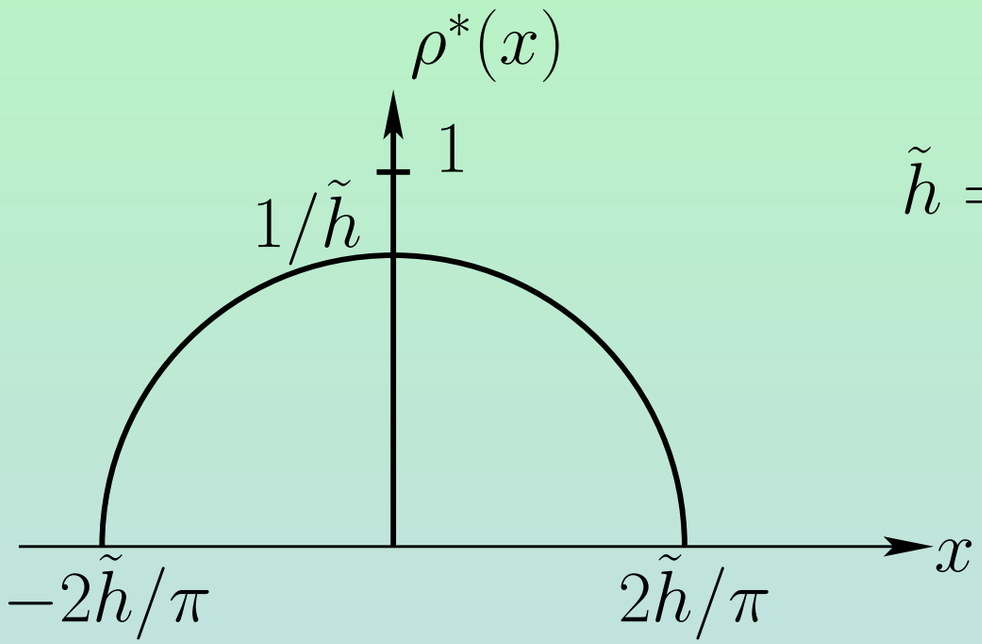
Large N asymptotics: saddle point analysis

- Douglas-Kazakov transition



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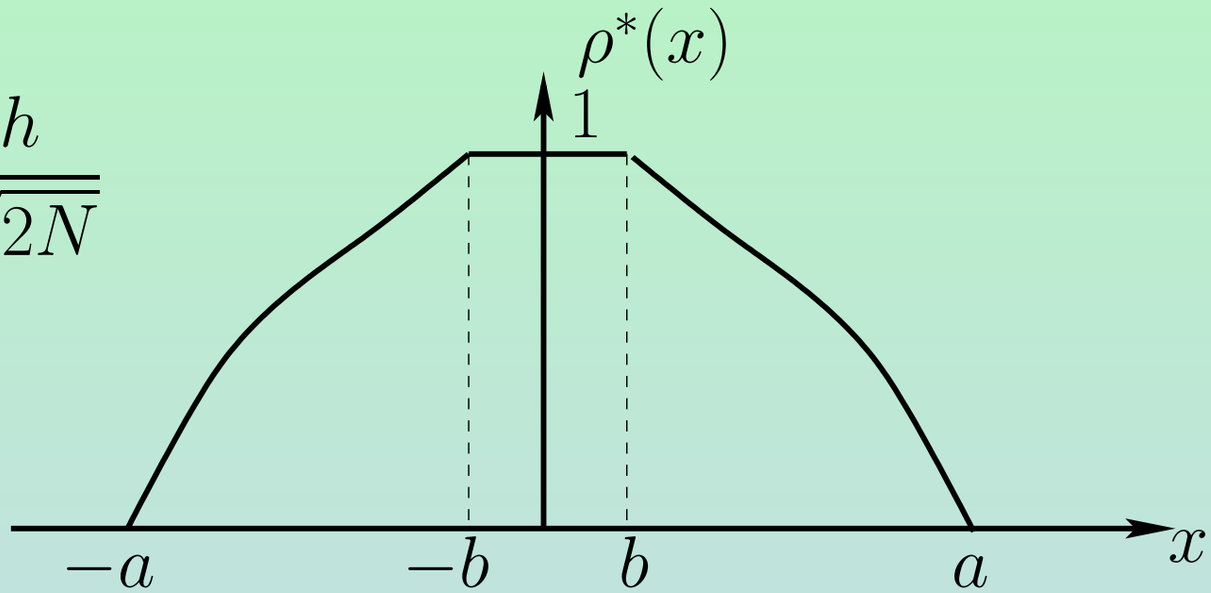

$\tilde{h} < 1$

- Third-order phase transition

$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N(\tilde{h}\sqrt{2N}) = \begin{cases} \phi^-(\tilde{h}) \,, \ \tilde{h} < 1 \,, \\ 0 \,, \ \tilde{h} \geq 1 \end{cases} \quad , \ \phi^-(\tilde{h}) \sim \frac{16}{3}(1-\tilde{h})^3 \,,$$

Large N asymptotics: explicit expression for the left tail

$$\phi_-(h) = 2\left(F_-(\pi^2/h^2) - F_+(\pi^2/h^2)\right)$$
$$F_-(X) = -\frac{3}{4} - \frac{X}{24} - \frac{1}{2}\ln X\ ,$$
$$F'_+(X) = \frac{a^2}{6} - \frac{a^2}{12}(1-k^2) - \frac{1}{24} + \frac{a^4}{96}(1-k^2)^2 X\ ,$$

where

$$k = \frac{b}{a}$$
$$a[2\mathbf{E}(k) - (1-k^2)\mathbf{K}(k)] = 1$$
$$aX = 4\mathbf{K}(k)$$

with

$$\mathbf{K}(y) = \int_0^1 \frac{dz}{\sqrt{1-y^2z^2}\sqrt{1-z^2}}\ ,\ \mathbf{E}(y) = \int_0^1 dz\frac{\sqrt{1-y^2z^2}}{1-z^2}$$

Large N asymptotics: orthogonal polynomials

$$F_N(h) \;=\; \frac{A_N}{h^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\le j<k\le N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^{N} n_i^2}$$

Use discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'} h_k \;,\; F_N(h) = \frac{\tilde{A}_N}{h^{2N^2+N}} \prod_{j=1}^{N} h_{2j-1}$$

Large N analysis of the three terms recursion relation

$$x\, p_k(x) = p_{k+1}(x) + R_k\; p_{k-1}(x) \;,\; R_k = \frac{h_k}{h_{k-1}}$$

Large N asymptotics: summary

P. J. Forrester, S. N. Majumdar, G. S., 2011

G. S., S. N. Majumdar, A. Comtet, P. J. Forrester, 2013

$$\begin{cases} F_N(h) & \sim \exp\left[-N^2\,\phi_-\left(h/\sqrt{2N}\right)\right]\,,\ h<\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(\sqrt{N}) \\ \\ 1-F_N(h) & \sim \exp\left[-N\,\phi_+\left(h/\sqrt{2N}\right)\right]\,,\ h>\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(\sqrt{N})\,, \end{cases}$$

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$$\phi_+(x) = 4x\sqrt{x^2-1} - 2\ln\left[2x\left(\sqrt{x^2-1}+x\right)-1\right]$$

Large N asymptotics: summary

P. J. Forrester, S. N. Majumdar, G. S., 2011

G. S., S. N. Majumdar, A. Comtet, P. J. Forrester, 2013

$$\begin{cases} F_N(h) & \sim \exp\left[-N^2\,\phi_-\left(h/\sqrt{2N}\right)\right]\,,\ h<\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(\sqrt{N}) \\ F_N(h) & \sim \mathcal{F}_1\left[2^{11/6}N^{1/6}(h-\sqrt{2N})\right]\,,\ h\sim\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(N^{-\frac{1}{6}}) \\ 1-F_N(h) & \sim \exp\left[-N\,\phi_+\left(h/\sqrt{2N}\right)\right]\,,\ h>\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(\sqrt{N})\,, \end{cases}$$

$\mathcal{F}_1$ is TW distribution for GOE

$$\mathcal{F}_1 = \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\,q^2(s)+q(s)\right)\,ds\right)$$

Describes the crossover through the 3rd order phase transition

Summary and conclusion

- Exact results for the large deviation functions of Gaussian ensembles of random matrices

$$\Pr.[\lambda_{\max} < w] = F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right] & , w < \sqrt{2} \,\&\, |w-\sqrt{2}| \sim \mathcal{O}(1) \\ \mathcal{F}_\beta\left(\sqrt{2} N^{\frac{2}{3}}(w-\sqrt{2})\right) & , \qquad |w-\sqrt{2}| \sim \mathcal{O}(N^{-\frac{2}{3}}) \\ 1-\exp\left[-\beta N \Phi_+(w)\right] & , w > \sqrt{2} \,\&\, |w-\sqrt{2}| \sim \mathcal{O}(1) \end{cases}$$

⟹ similar results for Wishart-Laguerre matrices for which the large deviations functions have been measured experimentally on coupled lasers (Fridman et al. '11)

- Associated third-order phase transition

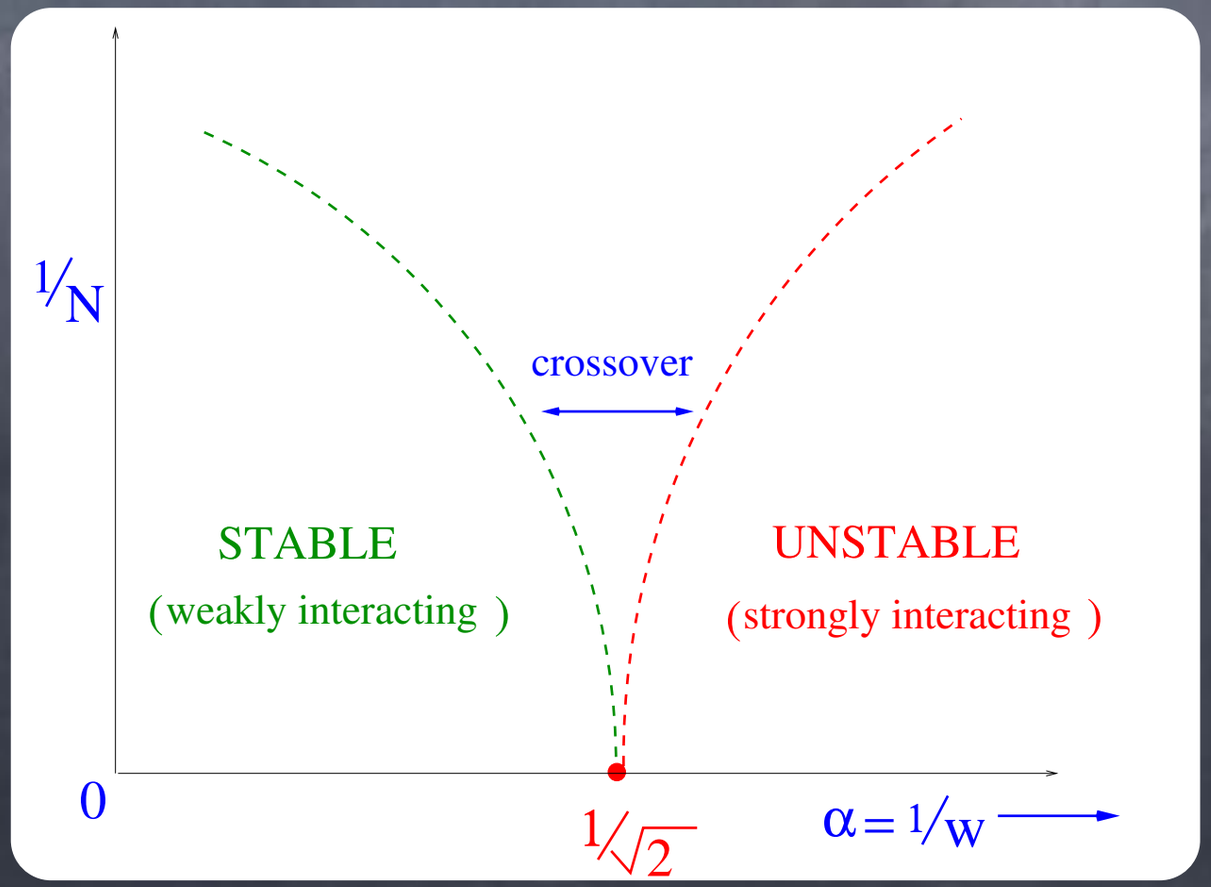
$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} \Phi_-(w) \propto (\sqrt{2}-w)^3 \,, & w < \sqrt{2} \,, \\ 0 & w > \sqrt{2} \end{cases}$$

Summary and conclusion

- Ubiquity in this third-order phase transition

  correlation matrices, 2d-gauge theory, non-intersecting Brownian motions, complexity in spin-glass models, random tillings...

- Tracy-Widom ditribution describes the crossover between the two regimes (at finite but large N)



ÉCOLE DE PHYSIQUE des HOUCHES

Session CIV

Université Joseph Fourier GRENOBLE

Stochastic processes and random matrices

July 6-31, 2015

http://lptms.u-psud.fr/workshop/randmat/

Organizing committee

Alexander Altland	Institut für Theoretische Physik, Universität Köln, Germany
Yan Fyodorov	School of Mathematical Sciences, Queen Mary University of London, Great-Britain
Neil O'Connell	Mathematics Institute, University of Warwick, Great-Britain
Grégory Schehr	Laboratoire de Physique Théorique et Modèles Statisitques, CNRS, Université Paris-Sud, France

Long Courses

Alexei Borodin (Massachusetts Institute of Technology)	Integrability and the Kardar-Parisi-Zhang universality class
Alice Guionnet (Massachusetts Institute of Technology)	An introduction to free probability
Pierre Le Doussal (CNRS, Ecole Normale Supérieure, Paris)	Replicas, renormalization and integrability in random systems
Satya Majumdar (CNRS, Université Paris-Sud, Orsay)	Recent applications of random matrices in statistical physics
Herbert Spohn (Universität München)	The Kardar-Parisi-Zhang equation – a statistical physics perspective
Bálint Virág (University of Toronto)	β-ensembles and random Schrödinger operators

Short Courses

Jean-Philippe Bouchaud (CFM, Ecole Polytechnique)	Random matrix theory and (big) data analysis
Alain Comtet (Université P. et M. Curie, Paris VI)	Schrödinger operators in random potentials
Bertrand Eynard (Commissariat à l'énergie atomique, Saclay)	Topological recursion in random matrices and combinatorics of maps
Jonathan Keating (School of Mathematics, University of Bristol)	Random matrix theory and number theory
Gernot Akemann (Universität Bielefeld)	Matrix models and quantum chromo-dynamics
Aris Moustakas (University of Athens)	Random matrix theory methods for telecommunication systems
Henning Schomerus (Lancaster University)	Random matrix theory approaches to open quantum systems
Vincent Vargas (CNRS, Ecole Normale Supérieure, Paris)	Gaussian multiplicative chaos and Liouville quantum gravity
Hans Weidenmüller (Universität Heidelberg)	Historical overview: random matrix theory and its applications
Anton Zabrodin (ITEP, Moscow)	Some aspects of integrability and quantum-classical correspondence