Large deviations and phase transitions in random matrix theory and related topics

Grégory Schehr
LPTMS, CNRS-Université Paris-Sud XI

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Collaborators:
- Alain Comtet (LPTMS, Orsay)
- Peter J. Forrester (Math. Dept., Univ. of Melbourne)
- Satya N. Majumdar (LPTMS, Orsay)
Main motivation: largest eigenvalue of random matrices

Recent excitements in statistical physics and mathematics on

$$\lambda_{\text{max}} = \max_{1 \leq i \leq N} \lambda_i : \text{largest eigenvalue of random matrices}$$

Gaussian random matrices belonging to Dyson’s ensembles $\text{GOE, GUE, GSE}$

Typical fluctuations (small):

✓ Tracy-Widom distributions
✓ ubiquitous/universal

KPZ equation, directed polymer, random permutation, correlation matrices, sequence alignment, ...

In this talk: atypical and large fluctuations of $\lambda_{\text{max}}$

✓ Large deviation functions
✓ Third order phase transition
Outline

1. Largest eigenvalue $\lambda_{\text{max}}$ of a Gaussian random matrix
   - Application to the stability of large complex system
     → Hint at a phase transition

2. Proba. distrib. func. (PDF) of $\lambda_{\text{max}}$ : Coulomb gas (CG) (with a wall)
   - Tracy-Widom distribution
   - Physics of large deviation tails
     → Third order phase transition

3. Related third order transition in other systems
   - Non-intersecting Brownian motions

4. Summary and conclusion
Stable non-interacting population of $N$ species with equilibrium densities $\rho_i^*$.

Slightly perturbed densities $x_i(t) = \rho_i(t) - \rho_i^*$ evolve via

$$\frac{dx_i(t)}{dt} = -x_i(t)$$

(assuming identical damping times)

Switch on interactions between the species

$$\frac{dx_i(t)}{dt} = -x_i(t) + \alpha \sum_{j=1}^{N} M_{i,j} x_j(t)$$

coupling strength

random interaction matrix

Q: what is the probability that the system remains stable once the interactions are switched on?
Linear stability criterion

Linear dynamical system

\[
\frac{d \mathbf{x}(t)}{dt} = (\alpha \mathbf{M} - \mathbb{I}) \mathbf{x}(t)
\]

Eigenvalues of \( \mathbf{M} \): \( \lambda_1, \lambda_2, \ldots, \lambda_N \)

The system is stable iff \( \alpha \lambda_i < 1, \quad \forall i = 1, 2, \ldots, N \)

i.e. iff \( \lambda_{\text{max}} < \frac{1}{\alpha} = w \)

Proba. that the system is stable \( P_{\text{stable}}(\alpha, N) \)

\[
P_{\text{stable}}(\alpha, N) = \text{Proba.}[\lambda_{\text{max}} < 1/\alpha = w]
\]
Stable/Unstable transition for large systems

Assuming that the interaction matrix is real, symmetric and Gaussian

\[ M_{i,j} = M_{j,i}, \quad P(M) \, dM \propto \exp \left[ -\frac{N}{2} \sum_{i,j} M_{i,j}^2 \right] \, dM = \exp \left[ -\frac{N}{2} \text{Tr}(M^2) \right] \, dM \]

May observed a sharp transition in the limit \( N \to \infty \)

\[
\lim_{N \to \infty} P_{\text{stable}}(\alpha, N) = \begin{cases} 
1, & \alpha < \alpha_c = 1/\sqrt{2} : \text{stable, weakly interacting phase} \\
0, & \alpha > \alpha_c = 1/\sqrt{2} : \text{unstable, strongly interacting phase}
\end{cases}
\]
Stable/Unstable transition for large systems

What happens for finite but large systems, of size $N \gg 1$?

$P_{\text{stable}}(\alpha, N) = \text{Prob}, [\lambda_{\text{max}} \leq w = 1/\alpha]$

- **LEFT TAIL**
- **STABLE**
- **RIGHT TAIL**
- **WEAK COUPLING**
- **STRONG COUPLING**

- Is there any thermodynamic sense to this transition?
- What is the analogue of the free energy?
- What is the order of this transition?
For large but finite $N$ : summary of the results

\[
P_{\text{stable}}(\alpha, N) = \text{Prob.}[\lambda_{\text{max}} \leq w = 1/\alpha]
\]

\[P_{\text{steady}} = \text{Prob.}[\lambda_{\text{max}} \leq w = 1/\alpha]
\]

\[
P_{\text{stable}}(\alpha, N) = \text{Prob.}[\lambda_{\text{max}} \leq w = 1/\alpha]
\]

\[
\Pr. [\lambda_{\text{max}} < w] = F_N(w) \approx \begin{cases} 
\exp \left[-\beta N^2 \Phi_-(w)\right] & , \ w < \sqrt{2} \& |w - \sqrt{2}| \sim \mathcal{O}(1) \\
F_\beta \left(\sqrt{2} N^{2/3} (w - \sqrt{2})\right) & , \ |w - \sqrt{2}| \sim \mathcal{O}(N^{-2/3}) \\
1 - \exp \left[-\beta N \Phi_+(w)\right] & , \ w > \sqrt{2} \& |w - \sqrt{2}| \sim \mathcal{O}(1)
\end{cases}
\]

and $\beta = 1$ (GOE) for May’s model

see also Mylène’s and Gaëtan’s talks

\[\checkmark \text{Crossover function: } F_\beta(x) \text{ Tracy & Widom ’94}\]

\[\checkmark \text{Large tail rate functions: } \Phi_\pm(w) \text{ Ben-Arous, Dembo & Guionnet ’01 , Dean & Majumdar ’06, Majumdar & Vergassola ’09}\]
Exact large deviation functions

Using Coulomb gas techniques and saddle point calculation for $N \gg 1$

Left deviation function

$$\Phi_-(w) = \frac{1}{108} \left[ 36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} \right.$$

$$\left. + 27 \left( \ln 18 - 2 \ln \left( w + \sqrt{w^2 + 6} \right) \right) \right], \ w < \sqrt{2}$$

when $w \to \sqrt{2}^-$

$$\Phi_-(w) \sim \frac{1}{6\sqrt{2}} (\sqrt{2} - w)^3, \ w \to \sqrt{2}^-$$

Dean & Majumdar ‘06

Right deviation function

$$\Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[ \frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right]$$

when $w \to \sqrt{2}^+$

$$\Phi_+(w) \sim \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2}, \ w \to \sqrt{2}^+$$

Ben-Arous, Dembo & Guionnet ‘01, Majumdar & Vergassola ‘09
This implies the behavior of the free energy

\[
\lim_{N \to \infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} 
\Phi_-(w) \propto (\sqrt{2} - w)^3, & w < \sqrt{2}, \\
0 & w > \sqrt{2}
\end{cases}
\]

The third derivative of the free energy is discontinuous

The crossover, for finite \( N \), between the two phases is described by the Tracy-Widom distribution

Third order phase transition

\[ \frac{1}{N} = \alpha \]

STABLE (weakly interacting)

UNSTABLE (strongly interacting)

crossover

\[ \frac{1}{\sqrt{2}} \]

\[ \alpha = \frac{1}{w} \]

Possible third-order phase transition in the large-$N$ lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 10 July 1979)

The large-$N$ limit of the two-dimensional U(N) (Wilson) lattice gauge theory is explicitly evaluated for all fixed $\lambda = g^2 N$ by steepest-descent methods. The $\lambda$ dependence is discussed and a third-order phase transition, at $\lambda = 2$, is discovered. The possible existence of such a weak- to strong-coupling third-order phase transition in the large-$N$ four-dimensional lattice gauge theory is suggested, and its meaning and implications are discussed.

$N = \infty$ PHASE TRANSITION IN A CLASS OF EXACTLY SOLUBLE MODEL LATTICE GAUGE THEORIES

Spenta R. WADIA
The Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Received 27 March 1980
Similar third order phase transition in gauge theory

Similar transition in $U(N)$ lattice gauge theory

Unstable phase = strong coupling phase of Yang–Mills gauge theory
Stable phase = weak coupling phase of Yang–Mills gauge theory

Tracy–Widom distribution describes the crossover between the two regimes (at finite but large $N$): double scaling regime
Outline

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  - Application to the stability of large complex system
    - Hint at a phase transition
  - Proba. distrib. func. (PDF) of $\lambda_{\text{max}}$: Coulomb gas (CG) (with a wall)
    - Tracy-Widom distribution
    - Physics of large deviation tails
      - Third order phase transition
  - Similar third order transition in other systems
    - Non-intersecting Brownian motions

- Summary and conclusion
Gaussian random matrix models

\( N \times N \) random matrix: \( \mathbf{M} \equiv M_{i,j} \)

Standard Dyson's ensembles: Orthogonal, Unitary, Symplectic

\( \text{(GOE)} \quad \text{(GUE)} \quad \text{(GSE)} \)

Gaussian probability measure \( P(\mathbf{M})d\mathbf{M} \propto \exp \left[ -\frac{\beta}{2} N \text{Tr}(\mathbf{M}^\dagger \mathbf{M}) \right] d\mathbf{M} \)

Joint PDF of the real eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_N \)

Wigner '51

\[
P_{\text{joint}}(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp \left[ -\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2 \right] \prod_{i<j} |\lambda_i - \lambda_j|^\beta
\]

\( \beta = 1 \text{(GOE)}, \quad \beta = 2 \text{(GUE)} \) and \( \beta = 4 \text{(GSE)} \)

Partition function

\[
Z_N = \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \exp \left[ -\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2 \right] \prod_{i<j} |\lambda_i - \lambda_j|^\beta
\]
Coulomb Gas picture and Wigner semi-circle

Rewrite the partition function as

$$Z_N(\beta) = \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \exp \left[ -\beta \left( \frac{N}{2} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \right) \right]$$

2-d Coulomb gas confined to a line, with $\beta$ the inverse temperature

Typical scale of the eigenvalues: $N^2 \lambda_{\text{typ}}^2 \sim N^2 \quad \longrightarrow \quad \lambda_{\text{typ}} \sim O(1)$

Mean density of eigenvalues

$$\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(\lambda - \lambda_i) \rangle$$

$$\quad \longrightarrow \quad \rho_{\text{SC}}(\lambda) = \frac{1}{\pi} \sqrt{2 - \lambda^2}$$
Tracy-Widom distributions for $\lambda_{\text{max}}$

In the limit $N \to \infty$, $\lambda_{\text{max}} \to \sqrt{2}$

The typical scale of fluctuations of $\lambda_{\text{max}}$ can be obtained via

$$\int_{\lambda_{\text{max}}}^{\sqrt{2}} \rho_{SC}(\lambda) \, d\lambda \sim \frac{1}{N} \quad \xrightarrow{\text{as } N \to \infty} \quad |\lambda_{\text{max}} - \sqrt{2}| \sim O(N^{-2/3})$$

Tracy-Widom distributions

$$\Pr[\lambda_{\text{max}} \leq w] \longrightarrow F_\beta(\sqrt{2}(w - \sqrt{2})N^{2/3}) \quad \text{Tracy & Widom '94}$$
Tracy-Widom distributions for $\lambda_{\text{max}}$

Asymmetric behavior of the PDF

$$\mathcal{F}_\beta'(x) \approx \begin{cases} 
\exp \left[ -\frac{\beta}{24} |x|^3 \right], & x \to -\infty \\
\exp \left[ -\frac{2\beta}{3} x^{3/2} \right], & x \to +\infty 
\end{cases}$$

Explicit expression in terms of a Painlevé II transcendent

Tracy & Widom '94

$$q''(x) = xq(x) + 2q^3(x), \quad q(x) \xrightarrow{x \to \infty} \text{Ai}(x)$$

Ubiquity of Tracy-Widom distributions

- largest eigenvalue of Wishart-Laguerre matrices
- longest increasing subsequence of random permutations
- directed polymers and growth models in the KPZ universality class
- continuum KPZ equation
- sequence alignment problems
- mesoscopic fluctuations in quantum dots
- maximal height of non-intersecting Brownian motions
Ubiquity of Tracy-Widom distributions

Experimental observation of TW distributions for GOE ($\beta = 1$) and GUE ($\beta = 2$) in liquid crystals experiments

Beyond TW distributions: large deviations of $\lambda_{\text{max}}$

TW-distributions describe the typical fluctuations, $|\lambda_{\text{max}} - \sqrt{2}| \sim O(N^{-2/3})$

What about the atypical fluctuations, $|\lambda_{\text{max}} - \sqrt{2}| \sim O(1)$?
Beyond TW distributions: large deviations of $\lambda_{\text{max}}$

\[ P_{\text{stable}}(\alpha, N) = \text{Prob.}[\lambda_{\text{max}} \leq w = 1/\alpha] \]

\[ w_c = \sqrt{2} \]

\[ w = 1/\alpha \]

\[ P_{\text{stable}}(\alpha, N) = \text{Prob.}[\lambda_{\text{max}} \leq w = 1/\alpha] \]

\[ 0 \]

\[ 1 \]

\[ \sqrt{2} \]

\[ -\sqrt{2} \]

\[ \rho(\lambda, N) \]

\[ \text{TRACY–WIDOM} \]

\[ N^{-2/3} \]

\[ \text{WIGNER SEMI–CIRCLE} \]

\[ \text{LEFT LARGE DEVIATION} \]

\[ \text{RIGHT LARGE DEVIATION} \]

\[ \text{LARGE DEVIATION} \]

\[ \text{LEFT TAIL} \]

\[ \text{RIGHT TAIL} \]

\[ \text{STABLE} \]

\[ \text{WEAK COUPLING} \]

\[ \text{STRONG COUPLING} \]

\[ 0 \]

\[ 1 \]

\[ \exp \left[-\beta N^2 \Phi_-(w)\right], \ \text{w < } \sqrt{2} \ \& \ |w - \sqrt{2}| \sim \mathcal{O}(1) \]

\[ \mathcal{F}_\beta \left(\sqrt{2} N^{2/3} (w - 2)\right), \ |w - \sqrt{2}| \sim \mathcal{O}(N^{-2/3}) \]

\[ 1 - \exp \left[-\beta N \Phi_+(w)\right], \ w > \sqrt{2} \ \& \ |w - \sqrt{2}| \sim \mathcal{O}(1) \]

\[ \text{UNSTABLE} \]

\[ \text{STABLE} \]

\[ \mathcal{O}(N^{-2/3}) \]

\[ \mathcal{O}(1) \]

\[ \text{LEFT TAIL} \]

\[ \text{RIGHT TAIL} \]

\[ \exp \left[-\beta N^2 \Phi_-(w)\right], \ \text{w < } \sqrt{2} \ \& \ |w - \sqrt{2}| \sim \mathcal{O}(1) \]

\[ \mathcal{F}_\beta \left(\sqrt{2} N^{2/3} (w - 2)\right), \ |w - \sqrt{2}| \sim \mathcal{O}(N^{-2/3}) \]

\[ 1 - \exp \left[-\beta N \Phi_+(w)\right], \ w > \sqrt{2} \ \& \ |w - \sqrt{2}| \sim \mathcal{O}(1) \]
Coulomb Gas with a wall
Cumulative distribution function of $\lambda_{\text{max}}$

$$
\Pr[\lambda_{\text{max}} \leq w] = \Pr[\lambda_1 \leq w, \lambda_2 \leq w, \ldots, \lambda_N \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)}
$$

$$
Z_N(w) = \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp \left[-\beta N^2 E[\{\lambda_i\}]\right]
$$

$$
E[\{\lambda_i\}] = \frac{1}{2N} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2N^2} \sum_{j \neq k} \log |\lambda_j - \lambda_k|
$$

What happens when the wall is moved?
Pushed vs. pulled Coulomb gas

\[ Z_N(w) = \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp \left[ -\beta N^2 E[\{\lambda_i\}] \right] \]

\[ E[\{\lambda_i\}] = \frac{1}{2N} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2N^2} \sum_{j \neq k} \log |\lambda_j - \lambda_k| \]

Saddle point analysis  
Dean & Majumdar ‘06, ‘08

Mean density of eigenvalues in presence of the wall \( \rho_w^*(\lambda) \)

\( \rho^*_w(\lambda) \)

\( w < \sqrt{2} \)
PUSHED

\( w = \sqrt{2} \)
CRITICAL

\( w > \sqrt{2} \)
PULLED
Left large deviation function

\[ F(w, N) = \text{Pr}[\lambda_{\text{max}} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim \exp[-\beta N^2 \Phi_-(w)], \ w < \sqrt{2} \]

i.e. \[ \lim_{N \to \infty} -\frac{1}{\beta N^2} \ln F(w, N) = \Phi_-(w) \]

Physically, \( N^2 \Phi_-(w) \) is the energy to push the Coulomb gas

Left deviation function

\[ \Phi_-(w) = \frac{1}{108} \left[ 36w^2 - w^4 - (15w + w^3)\sqrt{w^2 + 6} \right. \]

\[ + 27 \left( 18 - 2 \ln \left( w + \sqrt{w^2 + 6} \right) \right) \] , \( w < \sqrt{2} \)

when \( w \to \sqrt{2}^- \)

\[ \Phi_-(w) \sim \frac{1}{6\sqrt{2}} (\sqrt{2} - w)^3, \ w \to \sqrt{2}^- \]

Dean & Majumdar '06, '08
Matching between large deviation & and Tracy-Widom

when \( w \to \sqrt{2}^{-} \) \( \Phi_{-}(w) \sim \frac{1}{6\sqrt{2}} (\sqrt{2} - w)^3, \ w \to \sqrt{2}^{-} \)

matches with the left tail of the Tracy-Widom distribution

\[
\text{Pr}[\lambda_{\text{max}} \leq w] \sim \exp[-\beta N^2 \Phi_{-}(w) \sim] \sim \exp \left[ -\frac{\beta}{24} \left[ 2^{1/2} N^{2/3}(\sqrt{2} - w) \right]^3 \right]
\]

which coincides with the left tail of the TW distribution

\[
\mathcal{F}_\beta(x) \sim \exp \left( -\frac{\beta}{24} |x|^3 \right), \ x \to -\infty
\]
The saddle point equation yields a trivial result for \( w > \sqrt{2} \)

\[
F(w, N) = \Pr. [\lambda_{\text{max}} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim 1, \quad w > \sqrt{2}
\]

Non trivial corrections requires a different approach
Right tail: pulled Coulomb gas

\[
F(w, N) = \Pr[\lambda_{\text{max}} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)} \sim 1 - \exp(-N \Phi_+(w)) , \ w > \sqrt{2}
\]

\[\Phi_+(w) = \frac{1}{2} w \sqrt{w^2 - 2} + \ln \left[ \frac{w - \sqrt{w^2 - 2}}{\sqrt{2}} \right] \]

when \( w \to \sqrt{2}^+ \) \[\Phi_+(w) \sim \frac{2^{7/4}}{3} (w - \sqrt{2})^{3/2}, \ w \to \sqrt{2}^+ \]

\( N \Phi_+(w) \): energy to pull a single charge out of the Wigner sea

Ben-Arous, Dembo & Guionnet ’01, Majumdar & Vergassola ’09

matches with the right tail of the Tracy-Widom distribution
This implies the behavior of the free energy

\[
\lim_{N \to \infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} 
\Phi_-(w) \propto (\sqrt{2} - w)^3, & w < \sqrt{2}, \\
0, & w > \sqrt{2}
\end{cases}
\]

The third derivative of the free energy is discontinuous

The crossover, for finite \( N \), between the two phases is described by the Tracy-Widom distribution

Laguerre-Wishart random matrices

- Let $X$ be a $M \times N$ Gaussian random matrix (real or complex) with $M \geq N$.

- $W = X^\dagger X$ is a $N \times N$ symmetric or Hermitian Gaussian matrix.

- Joint probability density function of the $N$ real eigenvalues $\lambda_i$'s:

$$P(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} \prod_{i=1}^{N} \lambda_i^a \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N \frac{\beta}{2} \sum_{i=1}^{N} \lambda_i}, \ \lambda_i \geq 0, \ \forall i$$

where $\beta = 1$ (real) and $a = \frac{\beta}{2} (M - N + 1) - 1$ for $\beta = 2$ (complex).

- Largest eigenvalue in the Wishart-Laguerre ensemble:

$$\lambda_{\text{max}} = \max_{1 \leq i \leq N} \lambda_i$$
Largest eigenvalue of Laguerre-Wishart random matrices

\[ P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \prod_{i=1}^{N} \lambda_i^a \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} e^{-N \frac{\beta}{2} \sum_{i=1}^{N} \lambda_i}, \quad \lambda_i \geq 0, \quad \forall i \]

Density of eigenvalues: Marčenko-Pastur law

\[ \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \rightarrow_{N \to \infty} \rho_{MP}(\lambda) = \frac{1}{2\pi\lambda} \sqrt{(\lambda - \lambda_-)(\lambda_+ - \lambda)} \]

Large deviations of \( \lambda_{\text{max}} = \max_{1 \leq i \leq N} \lambda_i \):

\[ \Pr[\lambda_{\text{max}} < w] = F_N(w) \approx \begin{cases} 
\exp\left[-\beta N^2 \Psi_-(w)\right] & , w < \lambda_+ & |w - \lambda_+| \sim O(1) \\
\mathcal{F}_\beta\left(a_\beta N^{\frac{2}{3}}(w - \lambda_+)\right) & , |w - \lambda_+| \sim O(N^{-\frac{2}{3}}) \\
1 - \exp\left[-\beta N \Psi_+(w)\right] & , w > \lambda_+ & |w - \lambda_+| \sim O(1) 
\end{cases} \]

with \( \Psi_-(w) \overset{w \to \lambda_+}{\sim} (\lambda_+ - w)^3 \)

Majumdar & Vergassola '09
Vivo, Majumdar & Bohigas '07
Largest eigenvalue of Laguerre–Wishart random matrices in experiments

PHYSICAL REVIEW E 85, 020101(R) (2012)

Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers

Moti Fridman, Rami Pugatch, Micha Nixon, Asher A. Friesem, and Nir Davidson

Weizmann Institute of Science, Department of Physics of Complex Systems, Rehovot 76100, Israel
(Received 16 December 2011; published 1 February 2012)
A third order phase transition between a weak coupling and a strong coupling phase occurs when the gap, between the soft edge of a Coulomb droplet with density vanishing as a square-root at the edge and a hard wall at $w$, vanishes as one tunes a control parameter through its critical value.

Outline

- Largest eigenvalue $\lambda_{\text{max}}$ of a Gaussian random matrix
  - Application to the stability of large complex system
    - Hint at a phase transition
- Probability distribution function (PDF) of $\lambda_{\text{max}}$: Coulomb gas (CG) (with a wall)
  - Tracy-Widom distribution
  - Physics of large deviation tails
    - Third order phase transition
- Similar third order transition in other systems
  - Non-intersecting Brownian motions
- Summary and conclusion
Non-intersecting Brownian motions

- \( N \) non intersecting Brownian motions in one-dimension

\[
x_1(t) < x_2(t) < \ldots < x_N(t), \\
\forall t \geq 0
\]
Non-intersecting Brownian motions

- $N$ non intersecting Brownian motions in one-dimension

$$x_1(t) < x_2(t) < \ldots < x_N(t),$$
$$\forall t \geq 0$$

![Diagram of non-intersecting Brownian motions](image)
Non-intersecting Brownian motions

- $N$ non-intersecting Brownian motions in one-dimension

$$x_1(t) < x_2(t) < \ldots < x_N(t), \quad \forall t \geq 0$$

Tracy & Widom '07

watermelons

watermelons "with a wall"
An exact expression for the PDF of the maximal height

Distribution of the maximal height $\mathcal{H}_N$

$$F_N(h) = \mathbb{P} \left[ x_N(\tau) \leq h, \ \forall \ 0 \leq \tau \leq 1 \right]$$
An exact expression for the PDF of the maximal height

Distribution of the maximal height $H_N$

$$F_N(h) = \mathbb{P} [x_N(\tau) \leq h, \forall 0 \leq \tau \leq 1]$$

Exact result for finite $N$ G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling ’08

$$F_N(h) = \sum_{n_1, \ldots, n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1 \leq j < k \leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2} \sum_{i=1}^{N} n_i^2}$$

$$A_N = \frac{\pi 2N^2+N}{2^{N^2+N^2} \prod_{j=0}^{N-1} \Gamma(2+j)\Gamma(\frac{3}{2}+j)}$$

see also T. Feierl, M. Katori et al. ’08
Large $N$ asymptotics

- $F_N(h) \equiv$ partition function of a 2d gauge theory with symmetry group $\text{Sp}(2N)$, similar to the one studied by Douglas & Kazakov

  P. J. Forrester, S. N. Majumdar, G. Schehr ’11
Large $N$ asymptotics

- $F_N(h) \equiv$ partition function of a 2d gauge theory with symmetry group $Sp(2N)$, similar to the one studied by Douglas & Kazakov

- P. J. Forrester, S. N. Majumdar, G. Schehr ’11

- Saddle point analysis for large $N$, $h = \tilde{h} \sqrt{N}$

\[
F_N(h) \propto \sum_{n_1, \ldots, n_N = 0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1 \leq j < k \leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2 N} \sum_{i=1}^{N} n_i^2}
\]

when $N \to \infty$, $\frac{n_i}{2N} := x_i$ are continuous variables

\[
F_N(\tilde{h} \sqrt{2N}) \sim \int \mathcal{D} \tilde{\rho}(x) e^{-N^2 S[\tilde{\rho}]}, \quad \tilde{\rho}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i) \leq 2
\]

\[
S[\tilde{\rho}] = \frac{\pi^2}{\tilde{h}^2} \int_0^{a} dx \int_0^{a} dx' |\tilde{\rho}(x) - \tilde{\rho}(x')| \ln |x^2 - x'^2| + C \left[ \int_0^{a} \tilde{\rho}(x) dx - 1 \right]
\]
Large $N$ asymptotics: saddle point analysis

Constrained saddle point

\[
\int \mathcal{D}\tilde{\rho}(x)e^{-N^2S[\tilde{\rho}]} \sim \exp\left(-N^2S[\rho^*]\right), \quad \left. \frac{\delta S[\rho]}{\delta \tilde{\rho}(x)} \right|_{\tilde{\rho}=\rho^*} = 0
\]

\Rightarrow \text{integral equation for } \rho^*(x)

\[
\frac{\pi^2}{2h^2} x^2 - 2 \int_{-a}^{a} \rho^*(x') \ln |x - x'| dx' + C' = 0
\]

\[
\int_{-a}^{a} \rho^*(x) dx = 1, \quad \rho^*(x) \leq 1, \quad \forall x \in [-a, a]
\]
Large $N$ asymptotics: saddle point analysis

- Douglas-Kazakov transition

$$\tilde{h} > 1 \quad \text{and} \quad \tilde{h} < 1$$

$$\tilde{h} = \frac{h}{\sqrt{2N}}$$
Large $N$ asymptotics: saddle point analysis

- Douglas-Kazakov transition

\[ \tilde{h} = \frac{h}{\sqrt{2N}} \]

Third-order phase transition

\[
\lim_{N \to \infty} -\frac{1}{N^2} \ln F_N(\tilde{h} \sqrt{2N}) = \begin{cases} 
\phi^{-}(\tilde{h}), & \tilde{h} < 1, \\
0, & \tilde{h} \geq 1
\end{cases}
\]

, $\phi^{-}(\tilde{h}) \sim \frac{16}{3}(1 - \tilde{h})^3$, 

\[ \frac{1}{\tilde{h}} \]

\[ h > 1 \]

\[ h < 1 \]
Large $N$ asymptotics: explicit expression for the left tail

\[
\phi_-(h) = 2 \left( F_-(\pi^2/h^2) - F_+(\pi^2/h^2) \right)
\]

\[
F_-(X) = -\frac{3}{4} - \frac{X}{24} - \frac{1}{2} \ln X,
\]

\[
F'_+(X) = \frac{a^2}{6} - \frac{a^2}{12} (1 - k^2) - \frac{1}{24} + \frac{a^4}{96} (1 - k^2)^2 X,
\]

where

\[
k = \frac{b}{a}
\]

\[
a[2E(k) - (1 - k^2)K(k)] = 1
\]

\[
aX = 4K(k)
\]

with

\[
K(y) = \int_0^1 \frac{dz}{\sqrt{1 - y^2z^2}\sqrt{1 - z^2}}, \quad E(y) = \int_0^1 dz \frac{\sqrt{1 - y^2z^2}}{1 - z^2}
\]
Large $N$ asymptotics: orthogonal polynomials

\[ F_N(h) = \frac{A_N}{h^{2N^2} + N} \sum_{n_1, \ldots, n_N = 0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1 \leq j < k \leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2} \sum_{i=1}^{N} n_i^2} \]

Use discrete orthogonal polynomials

\[ \sum_{n=-\infty}^{\infty} p_k(n)p_{k'}(n)e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'} h_k \quad , \quad F_N(h) = \frac{\tilde{A}_N}{h^{2N^2} + N} \prod_{j=1}^{N} h_{2j-1} \]

Large $N$ analysis of the three terms recursion relation

\[ x p_k(x) = p_{k+1}(x) + R_k p_{k-1}(x) \quad , \quad R_k = \frac{h_k}{h_{k-1}} \]
Large \( N \) asymptotics: summary

P. J. Forrester, S. N. Majumdar, G. S., 2011

G. S., S. N. Majumdar, A. Comtet, P. J. Forrester, 2013

\[
\begin{align*}
F_N(h) & \sim \exp \left[ -N^2 \phi_- \left( \frac{h}{\sqrt{2N}} \right) \right], \quad h < \sqrt{2N} \text{ & } |h - \sqrt{2N}| \sim O(\sqrt{N}) \\
1 - F_N(h) & \sim \exp \left[ -N \phi_+ \left( \frac{h}{\sqrt{2N}} \right) \right], \quad h > \sqrt{2N} \text{ & } |h - \sqrt{2N}| \sim O(\sqrt{N}),
\end{align*}
\]
Large $N$ asymptotics: summary

P. J. Forrester, S. N. Majumdar, G. S., 2011
G. S., S. N. Majumdar, A. Comtet, P. J. Forrester, 2013

\[
\begin{cases}
F_N(h) & \sim \exp \left[ -N^2 \phi_+ \left( \frac{h}{\sqrt{2N}} \right) \right], \quad h < \sqrt{2N} \ & \ & |h - \sqrt{2N}| \sim O(\sqrt{N}) \\
1 - F_N(h) & \sim \exp \left[ -N \phi_+ \left( \frac{h}{\sqrt{2N}} \right) \right], \quad h > \sqrt{2N} \ & \ & |h - \sqrt{2N}| \sim O(\sqrt{N}),
\end{cases}
\]

\[\phi_+(x) = 4x\sqrt{x^2 - 1} - 2\ln \left[ 2x \left( \sqrt{x^2 - 1} + x \right) - 1 \right]\]
Large $N$ asymptotics: summary

P. J. Forrester, S. N. Majumdar, G. S., 2011
G. S., S. N. Majumdar, A. Comtet, P. J. Forrester, 2013

\[
\begin{align*}
F_N(h) & \sim \exp \left[ -N^2 \phi_-(h/\sqrt{2N}) \right], \quad h < \sqrt{2N} \text{ and } |h - \sqrt{2N}| \sim O(\sqrt{N}) \\
F_N(h) & \sim \mathcal{F}_1 \left[ 2^{11/6} N^{1/6} (h - \sqrt{2N}) \right], \quad h \sim \sqrt{2N} \text{ and } |h - \sqrt{2N}| \sim O(N^{-1/6}) \\
1 - F_N(h) & \sim \exp \left[ -N \phi_+ \left( h/\sqrt{2N} \right) \right], \quad h > \sqrt{2N} \text{ and } |h - \sqrt{2N}| \sim O(\sqrt{N}),
\end{align*}
\]

$\mathcal{F}_1$ is TW distribution for GOE

\[
\mathcal{F}_1 = \exp \left( -\frac{1}{2} \int_{-\infty}^{\infty} ((s-t)q^2(s) + q(s)) \, ds \right)
\]

Describes the crossover through the 3rd order phase transition
Summary and conclusion

Exact results for the large deviation functions of Gaussian ensembles of random matrices

$$\Pr[\lambda_{\text{max}} < w] = F_N(w) \approx \begin{cases} 
\exp\left[-\beta N^2 \Phi_-(w)\right], & w < \sqrt{2} \& |w - \sqrt{2}| \sim O(1) \\
F_\beta\left(\sqrt{2}N^{\frac{2}{3}}(w - \sqrt{2})\right), & |w - \sqrt{2}| \sim O(N^{-\frac{2}{3}}) \\
1 - \exp\left[-\beta N \Phi_+(w)\right], & w > \sqrt{2} \& |w - \sqrt{2}| \sim O(1)
\end{cases}$$

similar results for Wishart-Laguerre matrices for which the large deviations functions have been measured experimentally on coupled lasers (Fridman et al. ’11)

Associated third-order phase transition

$$\lim_{N \to \infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} 
\Phi_-(w) \propto (\sqrt{2} - w)^3, & w < \sqrt{2} \\
0 & w > \sqrt{2}
\end{cases}$$
Summary and conclusion

Ubiquity in this third-order phase transition correlation matrices, 2d-gauge theory, non-intersecting Brownian motions, complexity in spin-glass models, random tillings...

Tracy-Widom distribution describes the crossover between the two regimes (at finite but large $N$)
Stochastic processes and random matrices  
July 6-31, 2015  
http://lptms.u-psud.fr/workshop/randmat/  

Organizing committee

<table>
<thead>
<tr>
<th>Name</th>
<th>Institution</th>
</tr>
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<tbody>
<tr>
<td>Alexander Altland</td>
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</tr>
</tbody>
</table>

Long Courses

<table>
<thead>
<tr>
<th>Name</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alexei Borodin</td>
<td>Integrability and the Kardar-Parisi-Zhang universality class</td>
</tr>
<tr>
<td>Alice Guionnet</td>
<td>An introduction to free probability</td>
</tr>
<tr>
<td>Pierre Le Doussal</td>
<td>Replicas, renormalization and integrability in random systems</td>
</tr>
<tr>
<td>Satya Majumdar</td>
<td>Recent applications of random matrices in statistical physics</td>
</tr>
<tr>
<td>Herbert Spohn</td>
<td>The Kardar-Parisi-Zhang equation – a statistical physics perspective</td>
</tr>
<tr>
<td>Báltint Virág</td>
<td>β-ensembles and random Schrödinger operators</td>
</tr>
</tbody>
</table>

Short Courses

<table>
<thead>
<tr>
<th>Name</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jean-Philippe Bouchaud</td>
<td>Random matrix theory and (big) data analysis</td>
</tr>
<tr>
<td>Alain Comtet</td>
<td>Schrödinger operators in random potentials</td>
</tr>
<tr>
<td>Bertrand Eynard</td>
<td>Topological recursion in random matrices and combinatorics of maps</td>
</tr>
<tr>
<td>Jonathan Keating</td>
<td>Matrix models and quantum chromo-dynamics</td>
</tr>
<tr>
<td>Gernot Akemann</td>
<td>Random matrix theory methods for telecommunication systems</td>
</tr>
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<td>Aris Moustakas</td>
<td>Random matrix theory approaches to open quantum systems</td>
</tr>
<tr>
<td>Henning Schomerus</td>
<td>Gaussian multiplicative chaos and Liouville quantum gravity</td>
</tr>
<tr>
<td>Vincent Vargas</td>
<td>Historical overview: random matrix theory and its applications</td>
</tr>
<tr>
<td>Hans Weidenmüller</td>
<td>Some aspects of integrability and quantum-classical correspondence</td>
</tr>
<tr>
<td>Anton Zabrodin</td>
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