# Generating discrete-time constrained random walks

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## Motivations

Random walks appear in a wide range of phenomena ranging from ecology to finance. In many applications, one is interested in particular trajectories that satisfy some conditions. These trajectories are sometimes rare and atypical. One would like an efficient way to sample them.

## Free random walks

A free one-dimensional discrete-time random walk $x_m$ evolves according to the Markov rule

$$x_m = x_{m-1} + \eta_m \,, \tag{1}$$

where $\eta_m$ are *i.i.d.* random variables drawn from a jump distribution $f(\eta)$ and $x_0 = 0$.

## Propagators of free random walks

The probability density $P(x, m)$ that the free random walk is reaches $x$ in $m$ steps given that it started at the origin evolves according to the *forward* equation

$$P(x, m) = \int_{-\infty}^{\infty} dy\, P(y, m-1) f(x-y) \,, \tag{2}$$

with $P(x, 0) = \delta(x)$.
The probability density $Q(x, m)$ that the free random walk started at $x$ given that it reaches the origin in $m$ steps evolves according to the *backward* equation

$$Q(x, m) = \int_{-\infty}^{\infty} dy\, f(y-x) Q(y, m-1) \,, \tag{3}$$

with $Q(x, 0) = \delta(x)$.
The forward and backward equations can be solved explicitly by taking a Fourier transform. For the backward equation (3), we get

$$\tilde{Q}(k, m) = \hat{f}(k)\tilde{Q}(k, m-1) = \hat{f}(k)^m \,, \tag{4}$$

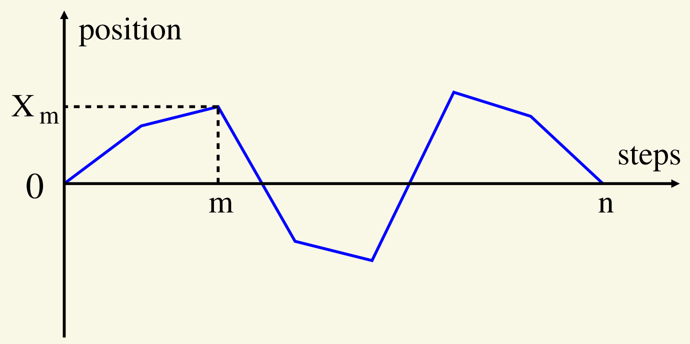
which after inversion gives

$$Q(x, m) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k)^m e^{-ikx} \,. \tag{5}$$

## Bridge random walks

Bridge random walks $X_m$ evolve locally as in (1) but are constrained to return to the origin after a fixed number of steps:

$$X_n = X_0 = 0 \,. \tag{6}$$




## Propagator of bridge random walks

The probability density $P_{\rm bridge}(X, m \,|\, n)$ that the bridge random walk of $n$ steps reaches $X$ in $m$ steps can be written as a simple product

$$P_{\rm bridge}(X, m \,|\, n) = \frac{P(X, m)\, Q(X, n-m)}{P(X=0, n)} \,, \tag{7}$$

where $P(X, m)$ accounts for the left part on $[0, m]$ and $Q(X, n-m)$ accounts for the right part on $[m, n]$.

## Generating bridge random walks

One can easily show that the bridge propagator (7) satisfies the forward equation

$$P_{\rm bridge}(X, m \,|\, n) = \int_{-\infty}^{\infty} dY\, P_{\rm bridge}(Y, m-1 \,|\, n) \times \tilde{f}(X - Y \,|\, Y, m-1, n) \,, \tag{8}$$

where the effective jump distribution is given by

$$\tilde{f}(\eta \,|\, Y, m-1, n) = f(\eta) \frac{Q(Y+\eta, n-m-1)}{Q(Y, n-m)} \,. \tag{9}$$

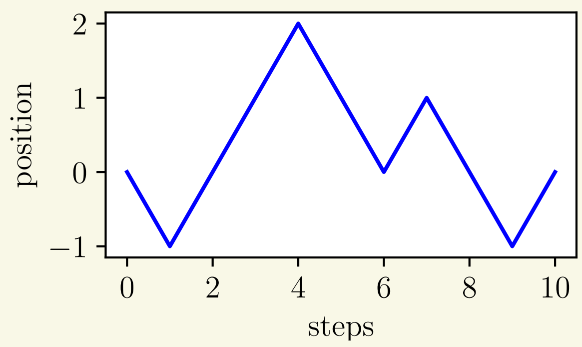
This effective jump distribution is well suited to be sampled using the acceptance-rejection method [1].

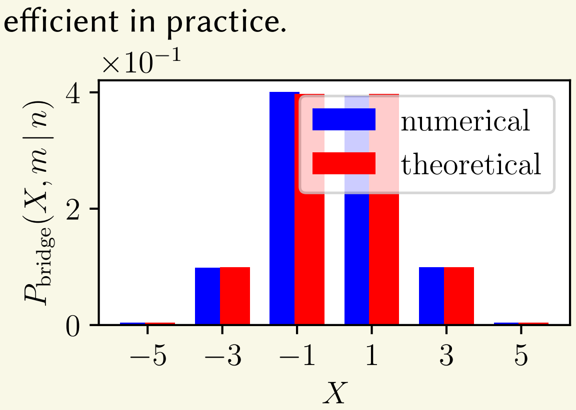
## Example: bridge lattice random walk

For a lattice walk, with $f(\eta) = \frac{1}{2}\delta(\eta-1) + \frac{1}{2}\delta(\eta+1)$, the effective jump distribution (9) becomes

$$\tilde{f}(\eta \,|\, Y, m-1, n) = \frac{1}{2}\left(1 - \frac{Y}{n-m}\right)\delta(\eta-1) + \frac{1}{2}\left(1 + \frac{Y}{n-m}\right)\delta(\eta+1) \,. \tag{10}$$

The effective distribution can be sampled directly and is shown to be very efficient in practice.




## Generalisations and future perspectives

The effective jump distribution (9) can be generalised to other constrained discrete-time random walks such as excursions and meander [1] as well as to some non-Markovian processes [2]. In a recent work [3], a reinforced learning approach was developed to generate rare atypical trajectories, with a given statistical weight and we hope that the method developed in our work will also be useful in such applications.

[1] B De Bruyne, S N Majumdar, G Schehr 2021 Generating discrete-time constrained random walks and Lévy flights arXiv:2104.06145.
[2] B De Bruyne, S N Majumdar, G Schehr 2021 Generating constrained run-and-tumble trajectories arXiv:2106.03385.
[3] D C Rose, J F Mair, J P Garrahan 2021 A reinforcement learning approach to rare trajectory sampling N. J. Phys.23 013013.