Generating constrained random walks

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Motivations

Random walks appear in a wide range of phenomena ranging from ecology to finance. In many applications, one is interested in particular trajectories that satisfy some conditions. These trajectories are sometimes rare and atypical. One would like an efficient way to sample them.

Free random walks

A free one-dimensional discrete-time random walk x_m evolves according to the Markov rule

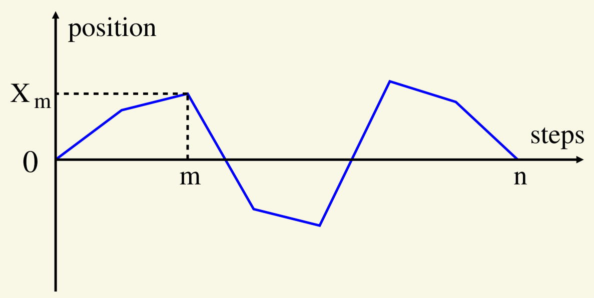
$$x_m = x_{m-1} + \eta_m\,, \tag{1}$$

where η_m are *i.i.d.* random variables drawn from a jump distribution $f(\eta)$ and $x_0 = 0$.

Bridge random walks

Bridge random walks X_m evolve locally as in (1) but are constrained to return to the origin after a fixed number of steps:

$$X_n = X_0 = 0\,. \tag{2}$$



Backward propagator of a free random walk

A useful tool is the probability density $Q(x, m)$ that the free random walk started at x given that it reaches the origin in m steps evolves according to the *backward* equation

$$Q(x, m) = \int_{-\infty}^{\infty} dy\, f(y-x) Q(y, m-1)\,, \tag{3}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k)^m e^{-ikx}\,. \tag{4}$$

with $Q(x, 0) = \delta(x)$.

Generating bridge random walks

One can easily show that the probability density $P_{\text{bridge}}(X, m \,|\, n)$ that the bridge random walk of length n is located at X at step m satisfies the forward equation

$$P_{\text{bridge}}(X, m \,|\, n) = \int_{-\infty}^{\infty} dY\, P_{\text{bridge}}(Y, m-1 \,|\, n) \times \tilde{f}(X - Y \,|\, Y, m-1, n)\,, \tag{5}$$

where the effective jump distribution is given by

$$\tilde{f}(\eta \,|\, Y, m-1, n) = f(\eta)\, \frac{Q(Y+\eta, n-m-1)}{Q(Y, n-m)}\,. \tag{6}$$

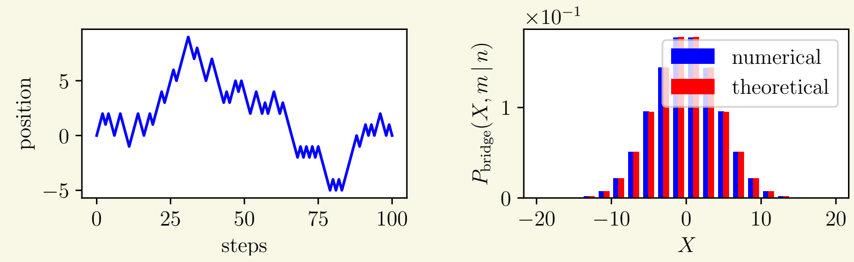
This effective jump distribution is well suited to be sampled using the acceptance-rejection method [1].

Example: bridge lattice random walk

For a lattice walk, with $f(\eta) = \frac{1}{2}\delta(\eta - 1) + \frac{1}{2}\delta(\eta + 1)$, the effective jump distribution (6) becomes

$$\tilde{f}(\eta \,|\, Y, m-1, n) = \frac{1}{2}\left(1 - \frac{Y}{n-m}\right)\delta(\eta - 1) + \frac{1}{2}\left(1 + \frac{Y}{n-m}\right)\delta(\eta + 1)\,. \tag{7}$$

The effective distribution can be sampled directly and is shown to be very efficient in practice.

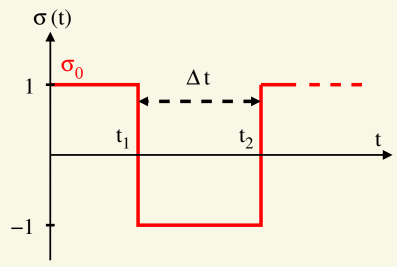


Generalisation to run-and-tumble particles

The position of a *free* run-and-tumble particle $x(t)$ evolves according to the Langevin equation

$$\dot{x}(t) = v_0\, \sigma(t)\,, \tag{8}$$

where $\sigma(t)$ is a telegraphic noise that switches between the values 1 and -1 with a *constant* rate γ:



The effective process, that automatically takes care of the bridge constraints

$$x(0) = 0\,, \quad \dot{x}(0) = \sigma_0\,, \quad x(t_f) = 0\,, \quad \dot{x}(t_f) = \sigma_f\,, \tag{9}$$

can be written as

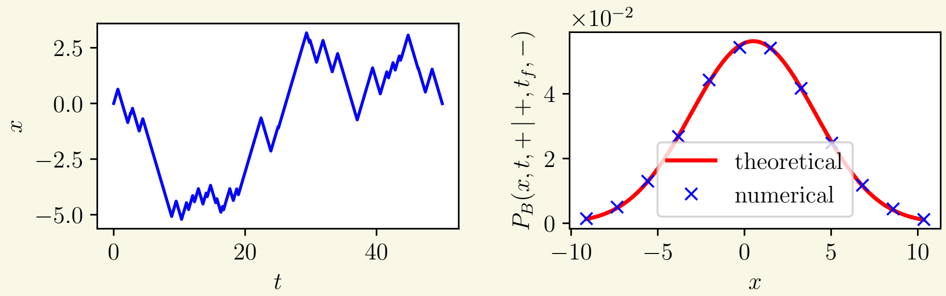
$$\dot{x}(t) = v_0\, \sigma^*(x, \dot{x}, t \,|\, \sigma_0, t_f, \sigma_f)\,, \tag{10}$$

where $\sigma^*(x, \dot{x}, t \,|\, \sigma_0, t_f, \sigma_f)$ is now an effective telegraphic noise that switches between the values 1 and -1 with space-time dependent rates

$$\gamma^*(x, \dot{x} = +v_0, t \,|\, \sigma_0, t_f, \sigma_f) = \gamma\, \frac{Q(x, t_f - t, - \,|\, \sigma_f)}{Q(x, t_f - t, + \,|\, \sigma_f)}\,, \tag{11}$$

$$\gamma^*(x, \dot{x} = -v_0, t \,|\, \sigma_0, t_f, \sigma_f) = \gamma\, \frac{Q(x, t_f - t, + \,|\, \sigma_f)}{Q(x, t_f - t, - \,|\, \sigma_f)}\,, \tag{12}$$

where Q is the backward propagator of a free run-and-tumble particle.



Generalisations and future perspectives

The effective jump distribution (6) and effective equation of motion (10) can be generalised to other constrained processes such as excursions and meanders [1,2], as well as non-intersecting walks [3]. In a recent work [4], a reinforced learning approach was developed to generate rare atypical trajectories, with a given statistical weight and we hope that the method developed in our work will also be useful in such applications.

[1] B De Bruyne, S N Majumdar, G Schehr 2021 Generating discrete-time constrained random walks and Lévy flights arXiv:2104.06145.
[2] B De Bruyne, S N Majumdar, G Schehr 2021 Generating constrained run-and-tumble trajectories arXiv:2106.03385.
[3] J Grela, S N Majumdar, G Schehr 2021 Non-intersecting Brownian bridges in the flat-to-flat geometry arXiv:2103.02545 .
[4] D C Rose, J F Mair, J P Garrahan 2021 A reinforcement learning approach to rare trajectory sampling N. J. Phys.23 013013.

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