Third excursion.

We shall revise the common belief of the spin-charge separation which seems to follow from exact solutions and is explicitly endorsed by the bosonization.

Intuitively, spin excitations must carry the electric current, as it takes place for free fermions, unless the switching on/off interactions is a singular limit.

The resolution comes from the effect of a collective current of vacuum states induced by spin excitations (in the Hubbard model approach) and in correct definition of current carrying states and current operators taking into account the band curvature (within the bosonization).

Related publication:
SPIN EXCITATIONS CARRY CHARGE CURRENTS: 1-D HUBBARD MODEL.
S. B., S. Matveenko & P. Nozieres, J. de Physique, 1994 ; S. M. & S. B. JETPh, 1994

Main approaches in theory of 1D interacting fermions:

Exact solutions , *Weak coupling* =

RG =g-ology - for fermions or with bosonization

Numerical methods (DMRG, etc.)

Bosonization method:

Great insight to physical properties in terms of phenomenological parameters of charge and spin density virtual sounds.

The manifestly spin − charge separation:

Hamiltonians are additive while typical operators, hence the correlation functions, become multiplicative among spin and charge variables.

Bethe Ansatz solution of the Hubbard model:

Describes the same system in terms of some particles of a deeper level (holons and spinons) supposed to correspond to ”deconfined” charge and spin degrees of freedom independently.

Free Fermi gas:

Elementary sound excitation is a shipment of a particle from a filled state **k**↑ to an empty one **(k+q)**↑↓
The corresponding charge current

$\mathbf{j = v_{k+q} - v_k =}$ $\partial v_k/\partial k$, $\mathbf{v_k = \partial\varepsilon/\partial k}$ - group velocity.

Spin ↑↓ does not matter:
same charge currents for singlet and triplet channels
Current is given by the spectrum curvature $\Gamma\mathbf{=}\partial\ \mathbf{v_k}\ /\partial\mathbf{k}$ –
a manifestation of the particle-hole non-symmetry.
j=0 at half-filling: $\Gamma\mathbf{=0}$ at the inflection point of $\varepsilon\mathbf{(k)}$.

Structure ofparticle states:
one quantum number $\mathbf{n_j}$ and the related quasi-momentum
$\mathbf{k_j=2\pi n_j/L}$ defines a state of one fermion
with both the charge **e** and the spin ½.

Bethe Ansatz (BA) exact solutions of the 1D Hubbard model for arbitrary interaction ***U*** and band filling ***ρ*** (*since Lieb & Wu, 1968*).

Different sets {M}, {N} of quantum numbers describe states of spin {M} and charge {N} sub-systems.

Excitations:

(Ovchinnikov 1970; Shiba 1972; Coll 1974, Woynarovich 1982, *…*
Building units: spinons or holons –
perturbations in {M} *or* {N} sequences

Added/extracted fermion splits into a spinon - holon pair with spectra $\boldsymbol{\varepsilon_s(q)}$ and $\boldsymbol{\varepsilon_h(q)}$

Spin-flip triplet excitation splits into two spinons –
similar to the spin-only model (Heisenberg spin chain).

Common conclusion: separation of spins and charges in a drastic contrast to free fermions, particularly the immanent charge current of spin excitations:
basic contradiction and motivation for our excursion.

$$H = -\sum_{n=1}^{N} (c_{n,\sigma}^{+} c_{n+1,\sigma} + h.c.) + u \sum_{n=1}^{N} c_{n,\uparrow}^{+} c_{n,\uparrow} c_{n,\downarrow}^{+} c_{n,\downarrow}$$

Unlike other excitations – holons and spin singlets, for spin triplets the spectra and the whole Bethe Ansatz construction evolve gradually from the Heisenberg chain equivalent at $\boldsymbol{\rho} = \mathbf{1}$ to arbitrary $\boldsymbol{\rho}$.
- tempting to be seen as spin waves even at $\boldsymbol{\rho} \neq \mathbf{1}$.
Also, a continuous evolution of spin excitations towards $\boldsymbol{u} = \mathbf{0}$ when they should become nothing but triplet electron-hole pairs, carrying hence the charge current at any $\boldsymbol{\rho} \neq \mathbf{1}$.

Ways to access the currents **j** of excited states:

1. Pray for getting its mean value –
   BA is not good for yielding values of operators.

2. Profit from the solution at presence of magnetic flux
Zvyagin & Krive , Shastry & Sutherland (1990-92).
*Energy of a ground or excited or an added state as functions of the flux **E(Φ)***

Currents of excitations in a Hubbard ring of N_0 cites with a flux magnetic flux **Φ** trough it.

$$\Phi \propto N_0\ ,\ \nu = \frac{2\pi}{N_0}\frac{\Phi}{\Phi_0}\ ,\ j = -N_0\frac{\partial E}{\partial \Phi}$$

Large ***u*** *limit + expansion in* ν **:**

the spinon with a small momentum $\mathbf{p_s}$

$$E = v_s p_s + 2\frac{\sin\pi\rho}{\pi}\nu^2 N_a + 2p_s\frac{\sin\pi\rho}{\pi\rho}\nu$$

$$v_s = \frac{2\pi}{u}\left(1 - \frac{\sin 2\pi\rho}{2\pi\rho}\right)$$

- small **~1/u** spin velocity,

$$j = -8(\Phi/\Phi_0)\sin(\pi\rho) + 2p_s\sin(\pi\rho)/\pi\rho$$

j = diamagnetic ground state contribution + paramagnetic charge current of the spinon.

$\mathbf{J_s=0}$ at ρ**=1** and $\mathbf{J_s\neq 0}$ even at $\rho\rightarrow$**0**

BA solution of Hubbard model on the ring of $\mathbf{N_0}$ sites or $\mathbf{N=\rho N_0}$ particles

with **M** spin ↓. GS: **M=N/2** , $\mathbf{S_z=0}$.

Any state is specified by two sets of integers: $\mathbf{I_j}$, $\boldsymbol{j = 1, .. N}$ and $\mathbf{J_\alpha}$, $\boldsymbol{\alpha = 1, .. M}$

Wave numbers quantization:

Scattering phase shifts: $\boldsymbol{\theta(\lambda) = -2\arctan(2\lambda/u)}$

1. Orbital numbers are shifted by scattering phases

$$N_0 k_j - \sum_{\beta=1}^{M} \theta\,(2\,sin\,k_j - 2\lambda_\beta) = 2\pi I_j = N_0 q_j$$

2. Spin wave numbers are composed only with phase shifts

$$\sum_{j=1}^{N} \theta\,(2\sin k_j - 2\lambda_\alpha) + \sum_{\beta=1}^{M} \theta\,(\lambda_\alpha - \lambda_\beta) = 2\pi J_\alpha = N_0 p_\alpha$$

Total momentum, energy, and current (fortunately !):

$$P = \sum_i k_i = \sum_i q_i + \sum_\alpha p_\alpha \quad E = -2\sum_i cos(\,k_i) \quad j = 2t\sum_i sin(k_i)$$

DIRECT ACCESS TO THE CURRENT

Origin of the charge current of a spin excitation:
At presence of the spin excitation, each $\mathbf{k}_i$ is shifted by an amount $\boldsymbol{\delta k_j}$ which is constant in the limit **u**>>1:

$$\delta k_i = \delta k = -(1/N_0)\sum_{\alpha} \theta\,(2\lambda_\alpha) = (1/N_0)\sum_{\alpha} p_\alpha = p_s/N_0$$

$\boldsymbol{p_s}$ - the total momentum of spin excitations.

The total charge current $\mathbf{j_s} = 2t\sum_i \cos(\mathbf{q}_i) = \mathbf{ps}\ /\mathbf{N_0}$

Looks like the GS energy, actually its second derivative – the curvature

Spinon does not perturb integer orbital numbers forming the GS.

But its appearance shifts the physical momenta $\boldsymbol{k_j}$,

thus giving rise to the orbital current.

The current is governed by the spinon momentum,

not by its velocity which is $\mathbf{v_s}$**~1/u** at **u>>1**.

Summary for spectra & current of a spinon and holon.
At small momenta **p= q-k$_F$** or over the whole zone **q**.
In limits of large (easy) and small
(difficult – praise Seguei Matveenko) interactions.

	spinon	Holon
U>>t	$j_s \approx 2p \sin(\pi\rho)/(\pi\rho)$ $\varepsilon_s \approx v_s p$, $v_s \sim 1/u$ j_s is detached from the velocity – no 1/u smallness	$j_h = 2\sin(\pi\rho) - 2\sin q$ $\approx 2p \cos(\pi\rho)$ $\varepsilon_h = 2\cos q - 2\cos(\pi\rho)$ $\approx 2p \sin(\pi\rho)$
U<<t	$\varepsilon_s = 2\cos q - 2\cos(\pi\rho/2)$ $\approx 2p \sin(\pi\rho/2)$ $j_s = 2\sin(\pi\rho/2) - 2\sin q$ $\approx 2p \cos(\pi\rho/2)$ resembles a single electron over the whole spectrum	$\varepsilon_h \approx 4\cos(q/2) - 4\cos(\pi\rho/2)$ $\approx 2p \sin(\pi\rho/2)$ $j_h \approx 2p \cos(\pi\rho/2)$ A pair of electrons at ½ of momenta

Always $j_s = 0$ @ $\rho = 1$ – half filling

Evolution of effective Fermi momentum of holons (not spinons) from the bare $k_F^0 = \pi\rho/2$ to the doubled $\pi\rho$ (spinless fermions)

Currents of eigenstates and of coherent states within the bosonization approach.

Weak coupling bosonization procedure - decomposition of the Fermi operator into R/L- moving parts $\boldsymbol{\Psi_{\sigma,\pm}}$,
spectrum linearization in the vicinity of $\boldsymbol{\pm k_F}$, interpretation of a two-parametric low energy cusp of particle-hole excitations as a single sound spectrum.

$$\boldsymbol{\Psi_\sigma(x) = \Psi_{\sigma+}(x)\exp(ik_F x) + \Psi_{\sigma-}(x)\exp(-ik_F x)}$$

$$\boldsymbol{H_0 = v_F(\Psi_{\sigma+}(-i\partial_x)\Psi_{\sigma+} - \Psi_{\sigma-}(-i\partial_x)\Psi_{\sigma-})}$$

$$\boldsymbol{\Psi_{\sigma,\pm} \propto exp\left[\frac{i}{2}[(\pm\varphi_\sigma - 4\pi\int_{-\infty}^{x}\pi_\sigma(x')dx')]\right], [\varphi_\sigma, \pi_\sigma] = -i\delta}$$

$$\boldsymbol{\Psi_\pm \propto exp\left[\frac{i}{2}[(\varphi_{gauge} \pm \varphi_{chiral}]\right], \ \partial_x\varphi_{gauge} \propto j, \partial_x\varphi_{chiral} \propto \rho}$$

φ = ($\varphi_{\uparrow}$+ $\varphi_{\downarrow}$) and ***σ* = (*σ* $_{\uparrow}$+ *σ* $_{\downarrow}$)** –
charge and spin phases = polarization fields
$\boldsymbol{\pi_{\varphi}}$ and $\boldsymbol{\pi_{\sigma}}$ - the conjugated momenta.

Charge density ***n*** and current ***j*** operators:

$$\boldsymbol{n} \propto \boldsymbol{\partial_x \varphi}, \quad \boldsymbol{j} = \boldsymbol{\Psi^{+} \sigma_z \Psi} \propto -\boldsymbol{\pi_{\varphi}} \propto -\boldsymbol{\partial_t \varphi}$$

contain the charge field operators only.
The eigenstates of ***H*(*σ*)** would carry no current,
would not interact with the electric field.

Repulsive case at ***ρ* ≠ 1**: two sounds

$$\boldsymbol{H} \Rightarrow \boldsymbol{H}(\boldsymbol{\varphi}) + \boldsymbol{H}(\boldsymbol{\sigma}),\ \boldsymbol{\varphi}, \boldsymbol{\sigma} = (\boldsymbol{\varphi}_{\uparrow} \pm \boldsymbol{\varphi}_{\downarrow})/\mathbf{2}$$

$$\boldsymbol{H}(\boldsymbol{\varphi}) = \boldsymbol{a}(\boldsymbol{\partial_x \varphi})^{\mathbf{2}} + \boldsymbol{b\pi_{\varphi}^{2}}\ ,\ \boldsymbol{H}(\boldsymbol{\sigma}) = \boldsymbol{c}(\boldsymbol{\partial_x \sigma})^{\mathbf{2}} + \boldsymbol{d\pi_{\sigma}^{2}}$$

***u* << 1**: taking into account the spectrum curvature
Γ ≈ cos(*πρ*/2) mixes the degrees of freedom.

$$\delta H = -\Gamma\Psi^{+}\partial_x^2\Psi \ , \ \delta j = \Gamma\Psi^{+}(-i\partial_x)\Psi \ , \ \delta n = 0$$

$$\delta H \sim (\partial_x\varphi)^3 + 3\partial_x\varphi\left[(\partial_x\sigma)^2 + \pi_\varphi^2 + \pi_\sigma^2\right] + 6\pi_\varphi\pi_\sigma\partial_x\sigma$$

$\partial_x\varphi$ - scalar invariant, allowed as a perturbation factor

$\pi_\varphi\pi_\sigma\partial_x\sigma$ = charge current × spin current × spin density
- nontrivial mixing invariant

$$\delta j = \sim -\Gamma(\pi_\varphi\partial_x\varphi + \pi_\sigma\partial_x\sigma) \ , \ \delta n = 0$$

Lagrangian expressions for **n** and **j** are intact –
work to keep the particles conservation law.
But the Hamiltonian relation $\partial_t\varphi = \pi_\varphi$ is elongated

$$n \propto \partial_x\varphi \ , \ j \propto -\partial_t\varphi \neq -\pi_\varphi$$

Spin excitations.

$a_k^\dagger, a_k$ **- magnon operators**

$$H_\sigma = \sum_k (cd)^{1/2} |k| a_k^\dagger a_k \ , \ \Omega = a_k^\dagger |0\rangle$$

$$\langle \Omega | j | \Omega \rangle = \langle \Omega | \delta j | \Omega \rangle = \Gamma k$$

current value is proportional to the momentum.

Charge excitations.

$$H_c = \sum_k (\mathrm{ab})^{1/2} |k| \mathrm{b}_k^\dagger b_k \qquad \Omega = b_k^\dagger |0\rangle$$

$$j_0 + \delta j = \langle \Omega | \pi_\varphi | \Omega \rangle + \Gamma \langle \Omega | \pi_\varphi \partial_x \varphi | \Omega \rangle = \langle \Omega | \delta j | \Omega \rangle = \Gamma k$$

Only the quasi-particle current from the holon, similar to spinion !
The expected collective current ~ phase velocity $\partial_t \varphi$
vanishes as given by the non-diagonal operator

$$\pi_\varphi = (b_k^\dagger - b_{-k})/\sqrt{k} \ , \ \langle \Omega | \pi_\varphi | \Omega \rangle = 0$$

For both the spinon and holon **$\langle j \rangle = \langle \delta j \rangle = \Gamma k$**
- just as was expected for free electrons

But where is the collective current - the only commonly expected one? $\boldsymbol{j_0 \sim \partial_t \varphi \sim \langle \pi_\phi \rangle}$

$j_0 \neq 0$ only for those coherent states
which are *not eigenstates*
of the charge sound Hamiltonian

1. Under a magnetic flux
2. In a coherent current carrying state, when the number of sounds bosons is not conserved due to the addition of the current controlling term in the Hamiltonian $\boldsymbol{j\Phi \sim \Phi\pi_\phi}$
3. High momentum mklapp excitation across Fermi points

$$\Omega_1 = \Psi_+^* \Psi_- \left|0\right\rangle \square \exp(i\varphi)\left|0\right\rangle, \; j \square \left\langle \pi_\varphi \right\rangle \square v_F$$

CONCLUSIONS. A revision of the spin – charge separation concept:

All elementary excitations of the Hubbard model
away from half filling carry the charge.

Spin excitations' currents are proportional to the momenta similarly to charge excitations.

The actual charge -spin separation emerges only at the level of a coherent current-currying ground state, when the number of charge bosons is not conserved and the current exists as their condensate.

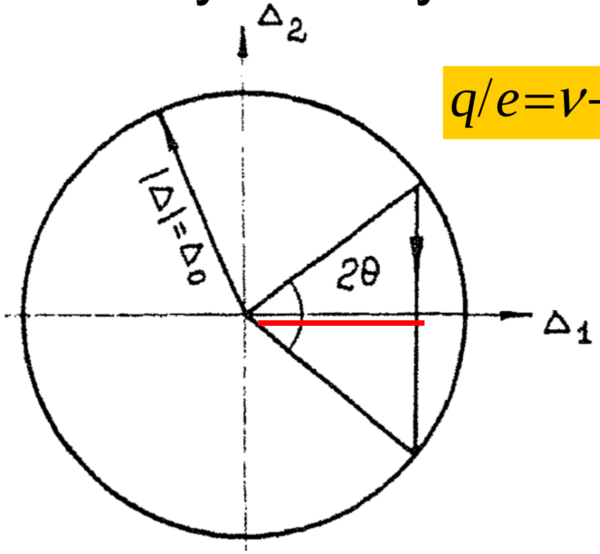
Contradiction to the spin-charge separation concept
as from the bosonization approach or from strong repulsion pictures.
But agreement with 1D free Fermi gas.

The paradox resolution and physical consequences:

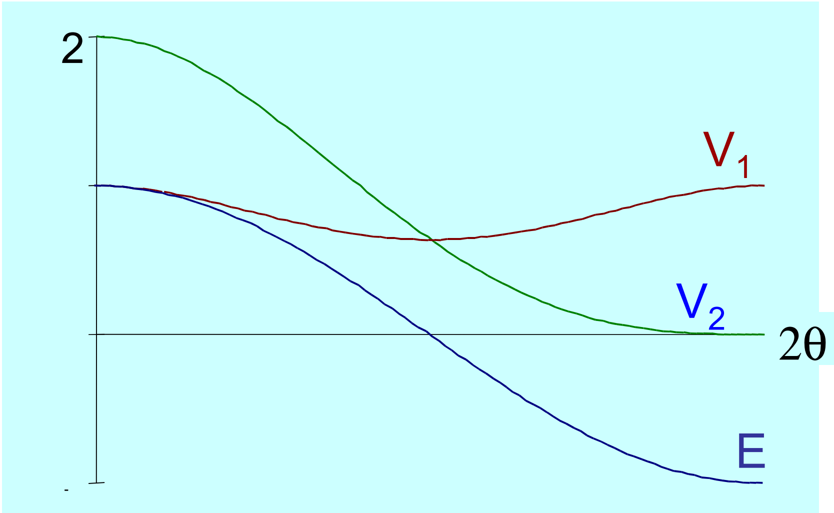
Weak coupling, Bosonization: account of Fermi velocity dispersion; reexamining of the current operator structure.

Strong coupling, BA for the Hubbard model:
Additional scattering phase shifts of vacuum holons in presence of an excited spinon.
A collective current in response to the single quasi-particle.

Degeneracy: family of chordus solitons in the complex plane of Δ



$q/e=\nu-\theta/\pi$



$$\Delta_1 = E_0 = \Delta_0 \cos\theta \quad k_0 = \Delta_0 \sin\theta$$

$$\Delta_2(x) = -k_0 \tanh(k_0 x)$$

$V_n(\theta)$ - selftrapping branches of total energy for chordus solitons with intragap state fillings n =1 and n=2.

The ν- filled spit-off state $\mathbf{E_0}$ is intragap but generally not the midgap !
Noninteger, variable charge **q**.
If no constraints on **θ**, the equillibrium solution for just one electron occupying the split off sate is **θ=π** hence **q=0**: particle with spin ½ and no charge – the spinon.

Sequence of chordus solitons develops from the bare **θ=0** through the amplitude soliton AS at **2θ=π** to the full phase slip **2θ=2π**.
Intra-gap split-off state E evolves from Δ_0 to $-\Delta_0$ providing the spectral flow across the gap together with the electrons' conversion.

$$\Delta = \Delta_1 + ik_0 \tanh(k_0 x) \qquad \xi_0 = \frac{\hbar v_F}{\Delta_0}$$

The single intragap state is at the midgap – "zero fermionic mode"

$$\Delta_1 = E_0 = \Delta_0 \cos\theta \quad k_0 = \Delta_0 \sin\theta$$

$$u_b(x) = \frac{\sqrt{k_0 / 2}}{\cosh(k_0 x)}$$

$$u_k = \frac{\Delta_0}{\sqrt{2L}} \frac{-k\xi_0 + ik_0 \tanh(x/\xi_0)}{E_k} e^{ikx}$$

Important features:

1. kink-like (topological soliton) shape of deformations.
2. Intragap, may not be mid-gap ,state at $\mathbf{E_0 \neq 0}$
3. Its wave function does contain the component which enters self-consisteny Eq.; hence the filling affects the equilibrium shape.
4. Delocalized states ($\mathbf{u_k}$) density is diluted which sums up to the compensating charge **0>q>-1**
5. These states show scattering phases which alone contribute to the soliton energy

Noninteger variable charge: sources and problems.

$$\rho_0(x) = \frac{k_0}{2\cosh^2(k_0 x)} \qquad \rho_k(x) = -\frac{\rho_0(x) k_0}{L\varepsilon_k(E_0+\varepsilon_k)} \qquad k_0 = \Delta_0 \sin\theta$$

$$\delta\rho(x) = \nu_0\rho_0 + \nu\sum_k \rho_k(x) = \left(\nu_0 - \nu\frac{\theta}{\pi}\right)\rho_0(x)$$

ν=2 – spin degeneracy of filled band states

ν_0 - filling of the split-off state.

Compensation of **ρ_0(x)** by local dilatations **~1/L** of L delocalized states.

Picture of a classical motion of the soliton **Δ(x-vt)**:

Fraction **1-2θ/π** of the charge **e** moves and gives the current,

Fraction **2θ/π** of the charge e is homogeneously distributed over the whole length **L**, it gives no current. Problem:

after quantization of the soliton, the whole complex of Bose field **Δ** and the fermions becomes a wave function distributed over **L**

– will the local and the delocalized charges recombine?

The total energy via its density

$$w_b(x) = -E_0\rho_0(x)$$

$$w_{lat} = \frac{\Delta^2(x)}{g^2} - \frac{\Delta_0^2}{g^2} = -\frac{k_0^2}{g^2\cosh(k_0 x)}$$

$$w_c(x) = -\sum_k \varepsilon_k \rho_k(x) = -\frac{\theta}{L\pi} E_0\rho_0(x)$$

$$w(x) = w_{lat} + \nu_0 w_b + \nu w_c = \Delta_0\rho_0(x)\left(\nu_0 - \nu\frac{\theta}{\pi}\right)\mathrm{Cos}\,\theta$$

The amplitude soliton, **θ=π/2** – the energy is zero! – wrong result. The same tric of phase shifts as for the real-field model recovers the missed term:

$$w(x) = \frac{\nu}{L\pi}\mathrm{Sin}\,\theta = \frac{2}{\pi}\Delta_0\frac{1}{L}$$

The energy $\mathbf{W_s = (2/\,\pi)\Delta} < \Delta$ is here, but it is totally delocalized; it does not move with the soliton.

Where does it all come from?