Two-Dimensional Coulomb Systems: a Larger Class of Solvable Models

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Abstract. - Using new methods, we find that a larger class of two-dimensional Coulomb systems are solvable models, in the framework of equilibrium classical statistical mechanics, for the special value \( \Gamma = 2 \) of the coupling constant. One-component plasmas with adsorption sites, one-component plasmas in a periodic background, and plasmas made of two components plus a background are discussed.

Matter is made of electrons and nuclei interacting through Coulomb’s law; therefore, it is obviously of interest to obtain exact theoretical results about Coulomb systems, and even “toy models” of them. The equilibrium statistical mechanics of several models of classical two-dimensional Coulomb systems has already been exactly worked out [1-5], at the special value \( \Gamma = 2 \) of the coupling constant \( \Gamma = \beta e^2 \), where \( \pm e \) is the charge of a particle and \( \beta \) is the inverse temperature (the Coulomb interaction potential between two particles of charge \( e \) at a distance \( r \) from one another is \( -\frac{e^2}{\ln(r/L)} \), where \( L \) is an arbitrary length scale). In this letter, we describe new methods which allow us to compute explicitly the \( n \)-body densities for a larger class of models, besides retrieving known results in a simpler and more systematic way.

For the one-component plasma OCP (a system of particles of charge \( e \) embedded in a continuous background of opposite charge), we are now able to deal with a variety of nonuniform backgrounds, and especially a background density having the periodicity of a two-dimensional crystal. At \( \Gamma = 2 \), the Boltzmann factor for \( N \) interacting particles with coordinates \( \mathbf{r}_i = (x_i, y_i) \) has the same structure in terms of a Slater determinant as the squared wave function of a system of noninteracting fermions, i.e.

\[
C|\det \{ \exp \left[ -V(\mathbf{r}_i) \right] z_{i,j}^{(N-1)} \}|_{i,j=1,...,N}^2,
\]

where \( C \) is a constant, \( e^2 V(\mathbf{r}_i) \) is the background potential acting on the \( i \)-th particle and \( z_i = x_i + iy_i \). For a constant background density \( \rho_b \), \( V(\mathbf{r}) \) can be chosen as \((1/2)\pi\rho_b r^2\), the functions \( \exp \left[ -V(\mathbf{r}) \right] z_i \) are mutually orthogonal; to deal with the Slater determinant is a standard problem, and it is easy to perform the integrals which define the \( n \)-particle densities and to take their thermodynamic limits [1]. In the general case of a nonuniform
background, the functions \( \exp[-V(r)]z^j \) are no longer orthogonal. However, we can follow the same steps as above in terms of a new basis \( \varphi_k(z) \) for the entire functions, chosen in such a way that the functions \( \varphi_k(r) = \exp[-V(r)]\varphi_k(z) \) are orthogonal, since the Slater determinant is invariant under such a change of basis. In terms of the projector

\[
\langle r_1 | P | r_2 \rangle = \sum_p \varphi_k(r_1)\bar{\varphi}_k(r_2)/\int \exp[\varphi_k(r)]^2,
\]

the \( n \)-particle-truncated densities are

\[
\begin{aligned}
\varphi(r) &= \langle r | P | r \rangle, \\
\varphi^2(r_1, r_2) &= -\langle r_1 | P | r_2 \rangle^2, \\
\varphi^{(n)}(r_1, r_2, \ldots, r_n) &= (-1)^n \sum_{i_1 \ldots i_n} \langle r_{i_1} | P | r_{i_2} \rangle \cdots \langle r_{i_n} | P | r_{i_1} \rangle,
\end{aligned}
\]

where the summation runs over all cycles \( (i_1 i_2 \ldots i_n) \) built with \( \{1, 2, \ldots, n\} \). Thus, the problem is reduced to computing the projector \( P \) on that subspace of Hilbert space which is spanned by the entire functions times \( \exp[-V(r)] \); this amounts to diagonalizing the matrix formed by the scalar products \( \int \exp[-2V(r)]z^j \).

As a first application, we can quickly retrieve all the known results [2] about the case where the background potential depends only on one coordinate: \( V(r) = V(x) \). Since \( V \) is translationally invariant along \( y \), it is convenient to choose the functions \( \varphi_k \) as \( \exp[-V(x) + k(x + iy)] \), with \( k \in \mathbb{R} \); they are indeed orthogonal because of the plane-wave factor \( \exp[iky] \). When the particles are confined to the half-space \( x > 0 \) by an impenetrable wall, the range of \( k \) must be restricted to \( k > 0 \), as it can be seen by reaching this case through a suitable limiting procedure.

When the background potential is periodic along \( y \), with a period \( b \), we can start with the same \( \varphi \)-functions \( \exp[k(x + iy)] \), writing \( k = 2\pi(\zeta + n)/b \), \( \zeta \in [0, 1] \), \( n \) integer; the scalar-product matrix is of the form

\[
\int dr \exp[2\pi(\zeta + n)(z/b) - 2V(r) + 2\pi(\zeta' + n)(z/b)] = \delta(\zeta - \zeta')A\zeta(n, n').
\]

Thus, as a second application, we are able to revisit a model for localized adsorption, which has been previously studied [3] by a tour de force of expansion resummations; we now obtain more general results in an easier way. The model is a line of equidistant adsorption sites located along the \( y \)-axis, creating a potential \( V_{\text{ads}} \) of the Baxter type, i.e. such that \( \exp[-\beta V_{\text{ads}}] = 1 + \lambda \delta(x) \sum_m \delta(y - mb) \). The continuous background density \( \rho_b(x) \) is assumed to depend only on \( x \), creating a potential chosen as \( V_0(x) \). Thus \( e^2V(r) = e^2V_0(x) + V_{\text{ads}} \). This is, for instance, a model for an electrode with adsorption sites. The matrix \( A\zeta(n, n') \) is found to be, up to a multiplicative constant, of the form \( \delta_{nn'} + f_\zeta(n)f_\zeta(n') \); thus, the diagonalization can be easily completed. We find

\[
\langle r_1 | P | r_2 \rangle = \exp[-V(r_1) - V(r_2)] \int d\zeta \sum_{nm} S(\zeta + n) \exp[2\pi(\zeta + n)z_\delta/b] \cdot \left\{ \frac{\delta_{nm} - \lambda \exp[-2V_0(0)]S(\zeta + m)}{1 + \lambda \exp[2V_0(0)]S(\zeta + l)} \right\} \exp[2\pi(\zeta + m)z_\delta/b],
\]
where $S$ is a normalization factor:

$$S(\zeta + n) = \left\{ b \int dx \exp \left[ 4\pi (\zeta + n) x/b \right] \exp \left[ -2V_0(x) \right] \right\}^{-1}.$$  

The summations on $n, m$, are on $\mathbb{Z}$ if the system occupies the whole plane, on $\mathbb{N}$ if the system is confined to the half-plane $x > 0$.

Lastly, we can deal with a background density having the double periodicity of a two-dimensional crystal. This model can be understood as made of mobile electrons interacting between themselves and with a lattice of extended fixed ions; this classical caricature of a metal (or perhaps of a ionic superconductor) has already been studied by computer simulation [6]. The background potential is of the form $e^2V(r) = e^2[V_0(x) + \phi(x, y)]$, where the potential $e^2V_0(x)$ created by the average background density $\rho_0$ can be taken such that $V_0(x) = \pi \rho_0 x^2$ and $e^2\phi(x, y)$ is a doubly periodical potential: $\phi(x + na, y + mb) = \phi(x, y)$; since there is one ion per lattice cell, $\rho_0 = (ab)^{-1}$. As a consequence of the symmetries of $V(r)$, an orthogonal basis is formed by the Bloch-type functions

$$\psi_\zeta (r) = \sum_n \exp [2\pi i \zeta n] [S(\zeta + n)]^{1/2} \exp [-V(r) + 2\pi (\zeta + n) z/b],$$

with $\zeta, \gamma \in [0, 1]$. The two-body correlation function is found to obey the Stillinger-Lovett sum rule [7] which characterizes a conducting phase. The detail will be published elsewhere; triangular lattices are also tractable. Here, we only quote a result for the simplest choice: a square lattice with $a = b = 1$, and $\exp [-2\phi] = 1 + \lambda \cos 2\pi x + \cos 2\pi y$. Then, the one-particle density is

$$\rho (r) = \rho_0 \sqrt{2} \exp [-2\phi] \int_0^1 d\zeta \int_0^1 d\gamma \frac{\exp [-\pi (x-\zeta)^2 - \pi (y-\gamma)^2 - 2i\pi (x-\zeta)(y-\gamma)]}{1 + \lambda \exp [-\pi/2] (\cos 2\pi x + \cos 2\pi y)}.$$

We are also able to generalize known results [5, 8-10] about the two-component plasma TCP (a system of positive and negative particles of charges $\pm e$). At $\Gamma = 2$, the TCP is equivalent to a free-fermion field. The system is unstable against collapse, unless some short-distance cut-off is introduced; however, if the cut-off is removed at constant fugacity, although the one-particle densities diverge, the $n$-particle–truncated densities ($n \geq 2$) have well-defined limits. We are able to consider a generalized TCP, made of positive particles, negative particles and a charged positive background. The background potential can be taken into account by introducing position-dependent fugacities $m_s(r) = m_s \exp [2\pi V(r)]$, where $s = +1(-1)$ if the particle at $r$ is positive (negative). The $n$-particle densities are again of the form (1), where now $\langle r_1 | P(r_2) \rangle$ is a $2 \times 2$ matrix in charge space (its $s_1 s_2$ element corresponds to a particle of charge $s_1 e\langle s_2 e \rangle$ at $r_1$($r_2$)); this matrix is no longer a projector but the Green function

$$\langle r_1 | P(r_2)_{s_1 s_2} = [m_{s_1}(r_1)]^{1/2} \langle r_1 \left[ \tau_x \partial_x + \tau_y \partial_y + \sum_{s = \pm 1} m_s(r) \frac{1 + s \tau_s}{2} \right]^{-1} r_2 \rangle_{s_1 s_2} [m_{s_2}(r_2)]^{1/2},$$

where $\tau_x, \tau_y, \tau_z$ are the Pauli matrices. These matrix elements can be re-expressed in terms of the isoscalar operators $A = \partial_x + i\partial_y + \partial_x V + i\partial_y V$ and $A^+ = -\partial_x + i\partial_y + \partial_x V - i\partial_y V$ as

$$\left\{ \begin{array}{l}
\langle r_1 | P(r_2) \rangle_{--} = \langle r_1 \left| m_+^2 [m^2 + A^+ A]^{-1} \right| r_2 \rangle, \\
\langle r_1 | P(r_2) \rangle_{++} = \langle r_1 \left| m_-^2 [m^2 + AA^+]^{-1} \right| r_2 \rangle, \\
\langle r_1 | P(r_2) \rangle_{+-} = -\langle r_2 \left| P(r_1) \right|_{-+} = -\langle r_1 \left| mA [m^2 + A^+ A]^{-1} \right| r_2 \rangle,
\end{array} \right.$$  

where $m^2 = m_+ m_-$. 

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In the case of a uniform background of charge density \( \varepsilon \), we can choose \( V(r) = (1/2) \pi \varepsilon r^2 \). The inversion of \( m^2 + A^+ A \) and \( m^2 + AA^- \) is easily done by solving a simple differential equation if we take \( r_2 \) at the origin; this is enough for computing \( \rho^{(2)}_T \) which depends only on \( r = |r_1 - r_2| \). We find

\[
\langle r | P | 0 \rangle_{-+} = \rho \Gamma (\alpha + 1) (\pi \varepsilon r^2)^{-1/2} W_{(1/2) - \alpha, 0} (\pi \varepsilon r^2),
\]

\[
\langle r | P | 0 \rangle_{+-} = \rho x \Gamma (\alpha + 1) (\pi \varepsilon r^2)^{-1/2} W_{-(1/2) - \alpha, 0} (\pi \varepsilon r^2),
\]

where \( \alpha = m^2/4\pi \varepsilon \) and \( W \) is a Whittaker function; the \((+-)\) and \((-+)\) matrix elements can be obtained by acting with the operator \( A \).

In the limit \( \varepsilon \to 0 \), we recover the usual TCP without a background. In the limit \( m_+ \to 0 \), the positive particles disappear and we are left with an OCP of negative particles in a positive background. In this limit, (2) becomes the projector on the solutions \( \phi \) of \( A \phi = 0 \), and these solutions are indeed of the form of an entire function of \( z = x + iy \) times \( \exp[-V] \). Thus, the OCP appears as a limiting case of our generalized TCP.

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