

We compute  $S(a) = \sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a)$

Using Poisson formula

$$1. \boxed{G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum_n c_n e^{i(2\pi n x - k x)} dx = \frac{1}{\sqrt{2\pi}} \sum_n c_n \underbrace{\int_{-\infty}^{+\infty} e^{i(2\pi n - k)x} dx}_{2\pi \delta(k - 2\pi n)}$$

$$= \sqrt{2\pi} \sum_n c_n \delta(k - 2\pi n)$$

2. we compute the  $c_n$  coefficients:

$$c_n = \int_{-1/2}^{1/2} \sum_{m=-\infty}^{+\infty} \delta(x-m) e^{-i2\pi n x} dx \quad \text{only } m=0 \text{ is in the interval}$$

$$= \int_{-1/2}^{1/2} \delta(x) e^{-i2\pi n x} dx = 1 \quad \text{so}$$

$$\boxed{G(k) = \sqrt{2\pi} \sum_{n=-\infty}^{+\infty} \delta(k - 2\pi n)}$$

is another Dirac comb.

3. We use Parseval-Plancherel equality:

$$\int_{-\infty}^{+\infty} f(x) g(x) dx = \int_{-\infty}^{+\infty} F(k) G(k) dk$$

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$$\int_{-\infty}^{+\infty} f(x) \sum_{n=-\infty}^{+\infty} \delta(x-n) dx = \sqrt{2\pi} \int_{-\infty}^{+\infty} F(k) \sum_{n=-\infty}^{+\infty} \delta(k - 2\pi n) dk$$

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$$\sum_n f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{+\infty} F(2\pi n)$$

4. we take  $\sigma = |a| > 0$ , then

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + a^2} = \frac{(\sqrt{2\pi})^2}{2\sigma} \sum_{n=-\infty}^{+\infty} e^{-2\pi\sigma|n|} = \frac{\pi}{\sigma} \left( 2 \underbrace{\sum_{n=0}^{\infty} (e^{-2\pi\sigma})^n}_{1/(1-e^{-2\pi\sigma})} - 1 \right) = \frac{\pi}{\sigma} \frac{2-1+e^{-2\pi\sigma}}{1-e^{-2\pi\sigma}}$$

$$= \frac{\pi}{\sigma} \frac{e^{\pi\sigma} + e^{-\pi\sigma}}{e^{\pi\sigma} - e^{-\pi\sigma}} = \frac{\pi}{\sigma} \coth(\pi\sigma)$$

as both sides work for  $a < 0$  (even function  $S(a) = S(-a)$ ) we obtain the result.

## Using the residue theorem

Bonus 8 along AB:  $z = N + \frac{1}{2} + it \quad t \in [-N - \frac{1}{2}, N + \frac{1}{2}]$

a) preliminary:

$$\begin{aligned} \cot \pi z &= i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \quad |\cot \pi z| = \frac{(e^{i\pi z} + e^{-i\pi z})(e^{-i\pi z^*} + e^{i\pi z^*})}{(e^{i\pi z} - e^{-i\pi z})(e^{-i\pi z^*} - e^{i\pi z^*})} \\ &= \frac{e^{i\pi(z-z^*)} + e^{-i\pi(z+z^*)} + e^{i\pi(z+z^*)} + e^{-i\pi(z-z^*)}}{e^{i\pi(z-z^*)} - e^{-i\pi(z+z^*)} - e^{i\pi(z+z^*)} + e^{-i\pi(z-z^*)}} \\ &= \frac{\cos(2\pi i \operatorname{Im}(z)) + \cos(2\pi \operatorname{Re}(z))}{\cos(2\pi i \operatorname{Im}(z)) - \cos(2\pi \operatorname{Re}(z))} \\ &\approx \frac{\operatorname{ch}(2\pi \beta) + \cos(2\pi \alpha)}{\operatorname{ch}(2\pi \beta) - \cos(2\pi \alpha)} \end{aligned}$$

if  $z = \alpha + i\beta$

b) Then, along AB,  $\cos(2\pi(N + \frac{1}{2})) = \cos \pi = -1$ , then

$$|\cot \pi z| = \frac{\operatorname{ch}(2\pi t) - 1}{\operatorname{ch}(2\pi t) + 1} \leq 1 - \frac{1}{\operatorname{ch}(2\pi t)} \leq 1$$

along BC,  $z = t + i(N + \frac{1}{2})$ , we get

$$\begin{aligned} |\cot \pi z| &= \frac{\operatorname{ch}(2\pi(N + \frac{1}{2})) + \cos(2\pi t)}{\operatorname{ch}(2\pi(N + \frac{1}{2})) - \cos(2\pi t)} \\ &= \frac{1 + \mu}{1 - \mu} \end{aligned}$$

since  $\mu = \frac{\cos(2\pi t)}{\operatorname{ch}(2\pi(N + \frac{1}{2}))}$  and  $|\cos| \leq 1 < \operatorname{ch}$  we have for  $N \geq 1$  that  $\mu \in [-1, 1]$  a priori.

but  $|\mu| < \frac{1}{\operatorname{ch}(2\pi(N + \frac{1}{2}))} < \frac{1}{\operatorname{ch}(\pi)}$  since  $\operatorname{ch}$  is an increasing function

finally  $|\cot \pi z| \leq \frac{1+|\mu|}{1-|\mu|} < \frac{1+1/\operatorname{ch}(\pi)}{1-1/\operatorname{ch}(\pi)} = \frac{\operatorname{ch}\pi + 1}{\operatorname{ch}\pi - 1} = K$  independent of  $N$   
 $\approx 1.188$

in conclusion

$$\boxed{\forall z \in \mathcal{Y}_N, |\cot \pi z| \leq K}$$

5. we have

$$\left| \oint_{\gamma_N} \pi \cot(\pi z) f(z) dz \right| \leq \pi \int_{\gamma_N} |\cot(\pi z)| |f(z)| dz \leq \pi K R \frac{L_N}{N^k} \xrightarrow[N \rightarrow \infty]{\text{length of } \gamma_N} \text{we have } |z| > N \text{ over } \gamma_N$$

$$\text{and } L_N = 4 \times (2N+1)$$

finally, we have

$$\leq \pi K R (8N^{1-k} + 4N^{-k}) \xrightarrow[N \rightarrow \infty]{} 0 \quad \text{since } k > 1$$

so that

$$\boxed{\oint_{\gamma_\infty} \pi \cot(\pi z) f(z) dz = 0}$$

6. we apply the theorem of residues to the function  $g(z) = \pi \cot(\pi z) f(z)$

the poles of  $g$  are split into the set  $\{z_j\}$ ,  $z_j \notin \mathbb{Z}$  and the poles coming from  $\cot \pi z$  which are  $z_n = n \in \mathbb{Z}$ , thus

$$\oint_{\gamma_\infty} \pi \cot(\pi z) f(z) dz = 0 = 2i\pi \left\{ \sum_{n \in \mathbb{Z}} \operatorname{Res}(g(z), z_n) + \sum_j \operatorname{Res}(g(z), z_j) \right\}$$

but  $\operatorname{Res}(g(z), z_n) = \frac{P(z_n)}{Q'(z_n)}$  for simple poles  
 $Q(z) = \sin(\pi z)$ ,  $Q'(z) = \pi \cos(\pi z)$   
 and  $\cos(\pi z_n) = \cos(\pi n) = \pm 1 \neq 0$

$$= \frac{\pi \cos(\pi n)}{\pi \cos(\pi n)} f(n) = f(n) ! \quad (\text{we understand the reason for the } \pi \text{ prefactor})$$

Eventually:

$$\boxed{\sum_{n \in \mathbb{Z}} f(n) = - \sum_j \operatorname{Res}(\pi \cot(\pi z) f(z), z_j)}$$

7. we guess we have to take  $f(z) = \frac{1}{z^2 + a^2}$  which satisfies to the condition: it has two poles  $z_j = \pm ia$ . Let's compute the residues: using the " $\frac{P(z_j)}{Q'(z_j)}$ " rule with  $Q(z) = z^2 + a^2$ , we have

$$S(a) = -\pi \cot(i\pi a) \frac{1}{2(i\pi a)} - \pi \cot(-i\pi a) \frac{1}{2(-i\pi a)}$$

then using that  $\cos(\pm i\pi a) = \cosh(\pm \pi a)$  and  $\sin(\pm i\pi a) = \pm i \sinh(\pi a)$ , we have  $\cot(\pm i\pi a) = \mp i \coth(\pi a)$  and

$$S(a) = + \frac{\pi}{2a} (\coth(\pi a) + \coth(-\pi a)) = \frac{\pi}{a} \coth(\pi a) \quad \text{Q.E.D.}$$

8. we understand we have to take properly the  $a \rightarrow 0$  limit:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} &= \frac{1}{2} \left( \sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + a^2} - \frac{1}{a^2} \right) \\ &\quad \text{( } \downarrow n=0 \text{ term)} \\ &= \frac{1}{2} \left( \frac{\pi}{a} \frac{\cosh(\pi a)}{\sinh(\pi a)} - \frac{1}{a^2} \right) = \frac{\pi a \cosh(\pi a) - \sinh(\pi a)}{2a^2 \sinh(\pi a)} \end{aligned}$$

$$\begin{aligned} &\stackrel{a \rightarrow 0}{\sim} \frac{\pi a (1 + \frac{1}{2}(\pi a)^2) - \pi a - \frac{1}{6}(\pi a)^3}{2a^2(\pi a)} \sim \frac{1}{2\pi} \left( \frac{1}{2} - \frac{1}{6} \right) \pi^3 = \frac{\pi^2}{6} ! \end{aligned}$$

10.a) we use again the transformation from  $\coth$  to  $\cot$ ,  $a$  is possibly complex in the formula, we set  $a = -it/\pi$ ,  $\coth(\pi a) = \coth(-it) = i \cot(t)$

$$\text{and } \coth(\pi a) = \frac{a}{\pi} \sum_{n \in \mathbb{Z}} \frac{1}{n^2 + (-it/\pi)^2} = -\frac{it}{\pi^2} \sum_{n \in \mathbb{Z}} \frac{1}{n^2 - t^2/\pi^2}$$

$$\text{so } \boxed{\cot(t) = t \sum_{n \in \mathbb{Z}} \frac{1}{t^2 - (n\pi)^2} = \frac{1}{t} + \sum_{k=1}^{\infty} \frac{2t}{t^2 - (k\pi)^2}}$$

$$\begin{aligned} b) \int_0^x (\cot(t) - \frac{1}{t}) dt &= \left[ \ln(\sin t - \ln t) \right]_0^x = \sum_{k=1}^{\infty} \int_0^x \frac{2t}{t^2 - (k\pi)^2} dt = - \sum_{k=1}^{\infty} \left[ -\ln((k\pi)^2 - t^2) \right]_0^x = \sum_{k=1}^{\infty} \ln \left( 1 - \frac{x^2}{(k\pi)^2} \right) \\ &\quad \text{to make the integral} \end{aligned}$$

taking the exponential gives the result.