Exam on Mathematical tools

3 hours

Wednesday January 9th

You are allowed to use only your notes and documents distributed during the lectures.
Do not mind about not doing everything, the exam is too long and the marking will take that into account.
*An estimate of points [**n**] for each question gives you an estimate of the length and difficulty of the answer. In no case it will correspond to the final grading, it is just here as a guide for you.*
Many parts and many of the questions are actually independent and can be done without solving the previous ones (but using intermediate results). Try to read and understand carefully the whole problem.
This exam uses the following topics: Gaussian integrals, steepest descend method, linear algebra, Green functions.

Ising model with infinite range interactions

We consider the Ising model where all spins interact with each others

$$H = -\frac{1}{2N}\sum_{i,j=1}^{N} \sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i \tag{1}$$

with variables $\sigma_i = \pm 1$ and N the number of spins, h the external field.

1. [**1**] Rewrite H using the variable $s = \sum_{i=1}^{N}\sigma_i$.
2. [**2**] Show that

$$e^{-\beta H(s)} = \sqrt{\frac{N\beta}{2\pi}}\int_{-\infty}^{\infty} dt\ \exp(-N\beta\frac{t^2}{2} + \beta(t+h)s) \tag{2}$$

3. [**2**] We introduce the partition function $Z = \sum_{\{\sigma_i=\pm 1\}} e^{-\beta H(s)}$. Show that

$$Z = \sqrt{\frac{N\beta}{2\pi}}\int_{-\infty}^{\infty} dt\ e^{-Nf(t)}\ ;\ \text{with } f(t) = \beta t^2/2 - \ln(2\cosh(\beta(t+h))). \tag{3}$$

4. [**3**] Perform a saddle point approximation on Z when $N \to \infty$. In particular determine the equation giving the position of the minimum (do not forget to discuss the sign of the second derivative). Interpret physically the meaning of the position of the saddle point and exhibit the solutions for $h = 0$.

Functional determinant

We consider the following one-dimensional linear differential operator of the form

$$\hat{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{4}$$

with $V(x)$ an arbitrary potential function ($\hat{H}$ is hermitian). We consider the following **boundary value problem**: for $x \in [0,1]$ with Dirichlet boundary conditions, the spectral problem for $\hat{H}$ reads

$$\hat{H}y(x) = Ey(x)\,, \quad \text{with } y(0) = y(1) = 0 \quad \text{and } E > 0\,, \tag{5}$$

in which E (energy) is the eigenvalue and $y(x)$ the corresponding eigenfunction.

The goal of this problem is to give a meaning to the function of the energy E corresponding to $\det(\hat{H} - E)$ which roots are clearly the eigenvalues of $\hat{H}$. In this case, it provides a way to give a meaning to the determinant of a differential operator. More precisely, we are going to discuss the **Gel'fand-Yaglom formula (1960)**

$$\frac{\det(\hat{H} - E\mathbb{I})}{\det(\hat{H})} = \frac{\phi(1,E)}{\phi(1,0)} \tag{6}$$

in which $\mathbb{I}$ is the identity operator and $\phi(x,E)$ is the solution, parametrized by E, of the following **initial value problem** (the prime stands for the first derivative with respect to x):

$$\hat{H}\phi(x) = E\phi(x)\,, \quad \text{with } \phi(0) = 0\,,\ \phi'(0) = 1 \quad \text{and } E > 0\,. \tag{7}$$

The discrete case

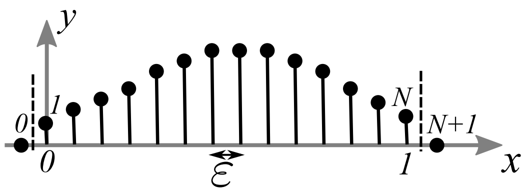

Figure 1: Discrete approach: N free variables y_j for $j = 1, \ldots, N$ while fixed ends means $y_0 = y_{N+1} = 0$.

In order to make a connection with linear algebra, we start with the discrete version of the problem. One can have in mind a chain of coupled oscillators. We divide the $[0,1]$ interval into $N-1$ elementary intervals of width $\varepsilon = 1/(N-1)$ and such that $y_j = y(x_j)$ with $x_j = (j-1)\varepsilon$ such that $x_1 = 0$, $x_N = 1$ (see figure 1). The **boundary value problem** corresponds to fixing the ends of the chain, i.e.

$$y_0 = y_{N+1} = 0\,. \tag{8}$$

We write $\mathbf{H}$ the matrix corresponding to the operator $\hat{H}$ and use the notations $V_j = \varepsilon^2 V(x_j)$, $\lambda = \varepsilon^2 E$.

1. [**2**] Why do we have the relation, for $j = 1, \ldots, N$: $y_{j+1} = (2 + V_j - \lambda)y_j - y_{j-1}$?
2. [**1**] Write down the structure of the $\mathbf{M}$ matrix such that $\mathbf{H} = \mathbf{M}/\varepsilon^2$ (of size $N \times N$).

Recurrence relation for the initial value problem

We introduce the solution $\phi_j(\lambda)$ of the following discrete **initial value problem** for an arbitrary energy λ

$$\mathbf{M}\vec{\phi}(\lambda) = \lambda\vec{\phi}(\lambda)\,, \quad \text{with } \phi_0(\lambda) = 0\,, \quad \phi_1(\lambda) = 1\,. \tag{9}$$

3. [**2**] Show that $\phi_{N+1}(\lambda)$ is a polynomial of order N in λ with leading term $(-\lambda)^N$.
4. [**2**] By invoking the uniqueness of the characteristic polynomial, show that one gets

   $$\det(\mathbf{M} - \lambda\mathbb{I}) = \phi_{N+1}(\lambda) \tag{10}$$

   i.e. one can compute the determinant function associated to the eigenvalue problem with an arbitrary V_j by solving the corresponding initial value problem.

Example in the free case $V(x) = 0$

We introduce the $\mathbf{M}$ matrix, of eigenvalues λ_n and eigenvectors $\vec{y}_n$

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \tag{11}$$

5. **Spectrum by brute force calculation**
   a) **[2]** By looking for solutions of the form $y_j = cr^j$ with c and r two constants, show that r can take two distinct values $r_\pm$ such that $r_+ = 1/r_-$.
   b) **[3]** Therefore, the general form for an eigenvector is $y_j = ar_+^j + br_-^j$ with a, b two constants. Show that

$$\lambda_n = 2(1 - \cos q_n)\ , \qquad y_{n,j} \propto \sin(q_n j) \quad \text{where} \quad q_n = \frac{n\pi}{N+1}\ ,\ n = 1, \ldots, N \tag{12}$$

6. **Spectrum using the recurrence approach**
   a) **[3]** By trying $\phi_j = ae^{iqj} + be^{-iqj}$, show that the solution of the initial value problem is

$$\phi_j(\lambda) = \frac{\sin(qj)}{\sin q} \tag{13}$$

   provided q is related to λ. Recover the spectrum $\{\lambda_n\}$. *Remember* $\sin(a+b) = \sin a \cos b + \sin b \cos a$.
   b) **[2]** Infer the following (not so trivial) formula

$$\prod_{n=1}^{N} \left(\cos q - \cos\left(\frac{n\pi}{N+1}\right)\right) = \frac{\sin(q(N+1))}{2^N \sin q} \tag{14}$$

The continuous case

The free case

We now admit that the relation (6) is valid and we apply this result in the free case in which $\hat{H}_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$:

7. **[2]** What are the eigenvalues and eigenvectors (up to a normalization constant) of $\hat{H}_0$ for the boundary value problem (5)?

8. **[2]** Give the explicit solution for the initial value problem (7).

9. **[2]** Infer the relation

$$\frac{\sin q}{q} = \prod_{n=1}^{\infty} \left(1 - \left(\frac{q}{n\pi}\right)^2\right) \tag{15}$$

Thouless' approach via Green function (1972)

We here present a way of deriving (6) from a Green function approach.

Green function of the problem: We consider the boundary value problem (5) with E as a parameter.

10. **[1]** Write the corresponding equation and boundary conditions satisfied by the Green function $G_E(x, x')$.

11. **[1]** Following the tutorial and lecture, recall the two continuity equations satisfied by $G_E(x, x')$ when $x \to x'$.

12. **[3]** Let $\phi_L(x)$ and $\phi_R(x)$ be two independent solutions of the two initial value problems $\hat{H}\phi(x) = E\phi(x)$ with $\phi_L(0) = 0$, $\phi'_L(0) = 1$ and $\phi_R(1) = 0$, $\phi'_R(1) = -1$. Show that the Green function is equal to

$$G_E(x, x') = \begin{cases} \frac{\phi_L(x)\phi_R(x')}{W(x')} & \text{if } x \leq x' \\ \frac{\phi_R(x)\phi_L(x')}{W(x')} & \text{if } x \geq x' \end{cases} \tag{16}$$

with $W(x') = \phi'_L(x')\phi_R(x') - \phi'_R(x')\phi_L(x')$ (Wronskian).

13. **[2]** Show that $W(x) = \phi_L(1) = \phi_R(0)$ is actually just a constant.

14. **[2]** In the free case, give the explicit expression of $\phi_{L/R}(x)$ and of the Green function.

Obtaining the determinant

15. **[2]** We notice that the solutions $\phi_{L/R}(x)$ actually depend on E but this was made implicit before for clarity. Considering this dependence, show that

$$(\hat{H} - E\mathbb{I})\frac{\partial \phi_L(x)}{\partial E} = \phi_L(x) \quad \text{with initial values } \frac{\partial \phi_L(0)}{\partial E} = \frac{\partial \phi'_L(0)}{\partial E} = 0 \tag{17}$$

16. **[3]** Check that the following expression is solution of Eq. (17):

$$\frac{\partial \phi_L(x)}{\partial E} = -\frac{\phi_L(x)}{\phi_L(1)} \int_0^x \phi_L(y)\phi_R(y)dy + \frac{\phi_R(x)}{\phi_R(0)} \int_0^x \phi_L^2(y)dy \tag{18}$$

17. **[2]** Deduce the following relation in which the dependence of ϕ_L is restored explicitly

$$\int_0^1 G_E(x, x)dx = -\frac{1}{\phi_L(1, E)} \frac{\partial \phi_L(1, E)}{\partial E} \tag{19}$$

18. **[2]** We recall that in the linear algebra tutorials, we obtained the formula $\frac{\mathrm{d}}{\mathrm{d}x} \det(\mathbf{A}) = \det(\mathbf{A}) \operatorname{Tr}\left(\mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\right)$ for $\mathbf{A}$ an Hermitian matrix. Motivate the following definition for the functional determinant

$$\frac{1}{\det(\hat{H} - E\mathbb{I})} \frac{\mathrm{d}}{\mathrm{d}E} \det(\hat{H} - E\mathbb{I}) \equiv -\operatorname{Tr} \hat{G}_E \tag{20}$$

in which $\hat{G}_E$ is the operator form of the Green function.

19. **[2]** Finally show that we have the Gel'fand-Yaglom relation (6) by assuming for sake of simplicity that the $\phi_L(1, E)$ and $\det(\hat{H} - E\mathbb{I})$ functions are positive.

The end.