Ising model with infinite range interactions

1. $H(s) = -s^2/(2N) - hs$.

2. One recognizes the result of a Gaussian integral with $a = \beta N$, $b = \beta s$ (notation of the lecture):

$$\int_{-\infty}^{\infty} dt\, e^{-N\beta\frac{t^2}{2}+\beta(t+h)s} = e^{\beta hs}\int_{-\infty}^{\infty} dt\, e^{-(N\beta)\frac{t^2}{2}+(\beta s)t} = e^{\beta hs}\sqrt{\frac{2\pi}{N\beta}}e^{(\beta s)^2/(2N\beta)} = \sqrt{\frac{2\pi}{N\beta}}e^{-\beta H(s)} \tag{1}$$

3. We introduce the partition function $Z = \sum_{\{\sigma_i=\pm1\}} e^{-\beta H(s)}$. We have the factorization

$$\sum_{\{\sigma_i=\pm1\}} e^{\beta(t+h)\sum_i\sigma_i} = \prod_{i=1}^{N}\sum_{\sigma=\pm1} e^{\beta(t+h)\sigma} = (2\cosh(\beta(t+h)))^N \tag{2}$$

so that

$$Z = \sqrt{\frac{N\beta}{2\pi}}\int_{-\infty}^{\infty} e^{-N\beta\frac{t^2}{2}}\sum_{\{\sigma_i=\pm1\}} e^{-N\beta\frac{t^2}{2}+\beta(t+h)s} = \sqrt{\frac{N\beta}{2\pi}}\int_{-\infty}^{\infty} e^{-N\beta\frac{t^2}{2}}(2\cosh(\beta(t+h)))^N = \sqrt{\frac{N\beta}{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-Nf(t)} \tag{3}$$

with $f(t) = \beta t^2/2 - \ln(2\cosh(\beta(t+h)))$.

4. It is the standard form for the saddle point approximation. We have $f'(t) = \beta(t - \tanh(\beta(t+h)))$, and $f''(t) = \beta(1-\beta(1-\tanh^2(\beta(t+h))))$. The saddle point thus satisfies to $t_c = \tanh(\beta(t_c+h))$ provided $f''(t_c) = \beta(1-\beta(1-t_c^2)) > 0$ (minimum). This is physically the self-consistent equation for the magnetization of the mean-field theory. For $h = 0$, the critical temperature corresponds here to $\beta = 1$. When $\beta < 1$, $t_c = 0$, clearly $f''(t_c) > 0$, $f(t_c) = -\ln 2$ and the saddle point reads

$$Z \simeq \sqrt{\frac{\beta}{f''(t_c)}}e^{-Nf(t_c)} = \sqrt{\frac{1}{1-\beta}}e^{N\ln 2} \tag{4}$$

When $\beta > 1$, there are two degenerate minima with $t_c \neq 0$ not easy to determine analytically. In this case, one can show that $f''(t_c) > 0$ and one has (the factor 2 sums up the minima contributions):

$$Z \simeq 2\sqrt{\frac{1}{1-\beta(1-t_c^2)}}e^{-N(\beta t_c^2/2-\ln(2\cosh(\beta t_c)))} \tag{5}$$

Thus, in the thermodynamical limit, the free energy of this model is (up to logarithmic corrections) equal to the mean-field result.

Functional determinant

1. Using $y_{j\pm1} = y(x_j \pm \varepsilon) \simeq y(x_j) \pm \varepsilon y'(x_j) + \frac{\varepsilon^2}{2}y''(x_j)$, one gets the usual discrete approximation of the Laplacian operator as $y''(x_j) \simeq (y_{j+1} + y_{j-1} - 2y_j)/\varepsilon^2$. The differential equation maps on $-(y_{j+1} + y_{j-1} - 2y_j)/\varepsilon^2 + (V_j/\varepsilon^2)y_j = (E/\varepsilon^2)y_j$ for $j = 1, \ldots, N$ using the boundary conditions $y_0 = y_{N+1} = 0$.

2. One has

$$\mathbf{M} = \begin{pmatrix} 2+V_1 & -1 & & \\ -1 & 2+V_2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2+V_N \end{pmatrix} \tag{6}$$

3. We have $\phi_{j+1} = (2+V_j-\lambda)\phi_j - \phi_{j-1}$ from the discrete recurrence relation. Furthermore, $\phi_0 = 0$, $\phi_1 = 1$, $\phi_2 = (2+V_1-\lambda)$, $\phi_3 = (2+V_2-\lambda)(2+V_1-\lambda) - 1$, etc... Clearly, the leading term in λ will be $\phi_j = (-\lambda)^{j-1} + \ldots$ are polynomials of order $j-1$ in λ and thus $(-\lambda)^N$ is the leading term in $\phi_{N+1}(\lambda)$.

4. Let λ_n be the N eigenvalues of $\mathbf{M}$, we have $\det(\mathbf{M} - \lambda\mathbb{I}) = \prod_{n=1}^{N}(\lambda_n - \lambda)$, which is the characteristic polynomial of the eigenvalue problem, with its leading term $(-\lambda)^N$. Furthermore, solutions of the boundary value problem can be obtained by tuning λ such that $\phi_{N+1}(\lambda) = 0$ (shooting method) so that $\vec{\phi}$ satisfies to the boundary conditions. Thus, the roots of $\phi_{N+1}(\lambda)$ are the eigenvalues λ_n. By invoking the uniqueness of the characteristic polynomial, we must have

$$\det(\mathbf{M} - \lambda\mathbb{I}) = \phi_{N+1}(\lambda) \tag{7}$$

Remark: another proof can be simply done by brute force calculation of the determinant $D_N = \det M$. After performing a transformation against the diagonal from bottom left to top right and expand the determinant against the first column, one gets the recursion relation $D_N = (2+V_N-\lambda)D_{N-1} - D_{N-2}$ which is the same as for the ϕ_j. It is initiated by $D_0 = 1$ and $D_1 = (2+V_1-\lambda)$ and we have finally $\phi_{N+1} = D_N$.

5. a) The recurrence relation is $y_{j+1} = (2-\lambda)y_j - y_{j-1}$. With the ansatz $y_j = cr^j$, it yields $r^2-(2-\lambda)r+1 = 0 = (r-r_-)(r-r_+)$. We write the two roots has $r_\pm$. If there are equal $r_- = r_+ = R \neq 0$, as for differential equation, the eigenvectors would take the form $y_j = aR^j + bjR^j$ with a and b two constants. But, $y_0 = 0 \Rightarrow a = 0$ and $y_{N+1} = 0 \Rightarrow b = 0$, which is absurd. Consequently, the roots are distinct, in particular, since $r_+r_- = 1$ from the last coefficient of the quadratic equation, we have $r_+ \equiv R$ and $r_- = 1/R$.

   b) The general form of an eigenvector is thus $y_j = aR^j + bR^{-j}$ with a and b two constants. $y_0 = 0 \Rightarrow a = -b$ and $y_{N+1} = 0 \Rightarrow R^{2(N+1)} = 1$. By consequence, the possible values are $R_n = e^{iq_n}$ with $q_n = \frac{n\pi}{N+1}$ with $n = 0, N+1$ discarded since one cannot have $|R| = 1$ ($r_+ \neq r_-$) so n takes only N distinct values $n = 1, \ldots, N$. The eigenvectors are then $y_{n,j} = 2ia\sin(q_nj)$ and from $2-\lambda_n = R_n + R_n^{-1} = 2\cos q_n$ so

$$\lambda_n = 2(1-\cos q_n)\,, \qquad y_{n,j} \propto \sin(q_nj) \tag{8}$$

6. a) First if we take $\phi_j = ae^{iqj} + be^{-iqj}$, we must have $\phi_0 = 0 = a+b$ and $\phi_1 = 1 = 2ia\sin q$, which gives $\phi_j = \frac{\sin(qj)}{\sin q}$. Now, we must find q such that this is a solution of the recurrence relation. By applying $\mathbf{M}$ on $\sin(qj)$, one gets $\mathbf{M}\sin(qj) = 2(1-\cos q)\sin(qj)$ so that $\lambda = 2(1-\cos q)$. Clearly, $\phi_{N+1}(q) = \frac{\sin(q(N+1))}{\sin q} = 0$ for $q = q_n$ (notice that $\phi_{N+1}(q=0) = N+1 \neq 0$ and $\phi_{N+1}(q=\pi) = (-1)^{N+1} \neq 0$ so we do recover $n = 1, \ldots, N$).

   b) We apply (7) with $\lambda_n = 2(1-\cos q_n)$, $\lambda = 2(1-\cos q)$ so $\det(\mathbf{M}-\lambda\mathbb{I}) = 2^N\prod_{n=1}^{\infty}(\cos q - \cos q_n)$ and $\phi_{N+1} = \frac{\sin(q(N+1))}{\sin q}$.

7. As in the lecture or basic quantum mechanics, we look for $y(x) = a\sin(qx) + b\cos(qx)$ solutions of the homogeneous equation. Boundary conditions imply that the eigenvectors are $y_n(x) = a_n\sin(q_nx)$ with $q_n = n\pi$, $n = 1, 2, \ldots, \infty$. Energies are simply $E_n = q_n^2 = (n\pi)^2$ which corresponds to the $q \to 0$ limit of $2(1-\cos q)$. Remark: the choice of sin/cos basis comes from the fact that energies must be positive: by writing $-\int y''y = E\int y^2 = \int y'^2 + [-y'y]_0^1$, one gets $E > 0$ for Dirichlet boundary conditions.

8. The differential equation is $-\phi''(x) = E\phi(x)$ of solutions $\phi(x) = a\sin(qx) + b\cos(qx)$ and we have $\phi(0) = 0 = b$ and $\phi'(0) = aq = 1$ that yields $\phi(x,E) = \sin(qx)/q$ ($q \neq 0$). From the equation, the parametrization is simply $E = q^2 > 0$.

9. We observe that writing $\det(\hat{H}-E\mathbb{I}) = \prod_{n=1}^{\infty}(E_n - E)$ leads to diverging infinite products (for instance $\det(\hat{H}_0 - E\mathbb{I}) = \prod_{n=1}^{\infty}((n\pi)^2 - E)$. A way to cope with this is to use the following ratio version of the formula

$$\frac{\det(\hat{H}-E\mathbb{I})}{\det(\hat{H})} = \frac{\phi(1,E)}{\phi(1,E=0)} \tag{9}$$

   with $\phi(1,E) = \sin q/q$, $\phi(1,E=0) = 1$ and $\frac{\det(\hat{H}-E\mathbb{I})}{\det(\hat{H})} = \prod_{n=1}^{\infty}((n\pi)^2 - q^2)/\prod_{n=1}^{\infty}(n\pi)^2$, we get (this relation can be checked by other means)

$$\prod_{n=1}^{\infty}\left(1 - \left(\frac{q}{n\pi}\right)^2\right) = \frac{\sin q}{q} \tag{10}$$

10. One rewrites the boundary value problem

$$(\hat{H}-E)G_E(x,x') = \delta(x-x')\,, \qquad \text{with } \forall x'\ G_E(0,x') = G_E(1,x') = 0 \tag{11}$$

11. As in the lecture, continuity and integrating the Dirac term provide the following two conditions (prime is the derivative with respect to x, x' being a parameter): for $\varepsilon \to 0^+$

$$G_E(x'+\varepsilon,x') = G_E(x'-\varepsilon,x') \text{ and } G_E'(x'-\varepsilon,x') - G_E'(x'+\varepsilon,x') = 1 \tag{12}$$

12. When $x < x'$, a solution of the homogeneous condition satisfying to the left boundary condition is $G_E(x,x') = A\phi_L(x)$ with A a constant that has to be set by (12). When $x' < x$, the same applies with the right boundary conditions $G_E(x,x') = B\phi_R(x)$. As in the tutorial, expliciting (12) provides the following linear system for A and B

$$\begin{pmatrix} \phi_L(x') & -\phi_R(x') \\ \phi_L'(x') & -\phi_R'(x') \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13}$$

   This system has non-zero solution provided its determinant $W(x') = \phi_L'(x')\phi_R(x') - \phi_R'(x')\phi_L(x')$ is non zero in which case one gets

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{W(x')}\begin{pmatrix} -\phi_R'(x') & \phi_R(x') \\ -\phi_L'(x') & \phi_L(x') \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow A = \frac{\phi_R(x')}{W(x')}\,,\ B = \frac{\phi_L(x')}{W(x')} \tag{14}$$

   and the proposed result for $G_E(x,x')$.

13. We have $W(x) = \phi_L'(x)\phi_R(x) - \phi_R'(x)\phi_L(x)$ so that $W' = \phi_L''\phi_R - \phi_R''\phi_L$ and using $\phi_{R/L}'' = (V-E)\phi_{R/L}$, one gets $W' = 0$ so $W(x)$ is constant $W(x) = W(0) = \phi_R(0)$ or $W(x) = W(1) = \phi_L(1)$.

14. We already found $\phi_L(x) = \sin(qx)/q$ in question 8. The same analysis gives $\phi_R(x) = \sin(q(1-x))/q$. We have $W = \sin q/q$ and finally

$$G_E(x,x') = \begin{cases} \frac{\sin(qx)\sin(q(1-x'))}{q\sin q} & \text{if } x \le x' \\ \frac{\sin(qx')\sin(q(1-x))}{q\sin q} & \text{if } x \ge x' \end{cases} \tag{15}$$

which is the result obtained in the tutorial.

15. As ϕ_L is solution of the homogeneous equation $(\hat{H} - E\mathbb{I})\phi_L(x) = 0$ and that $\frac{\partial}{\partial E}\left(\hat{H} - E\mathbb{I}\right) = -\mathbb{I}$, taking the partial derivative of this equation with respect to E yields:

$$(\hat{H} - E\mathbb{I})\frac{\partial \phi_L(x)}{\partial E} - \phi_L(x) = 0 \tag{16}$$

In addition, the boundary conditions at $x = 0$ are always the same and independent of E so $\frac{\partial \phi_L(0)}{\partial E} = \frac{\partial \phi'_L(0)}{\partial E} = 0$.

16. Let $f(x) = \phi_L(1)\frac{\partial \phi_L(x)}{\partial E}$ (remember that $\phi_L(1) = \phi_R(0)$). Then, using $\phi''_{R/L} = (V - E)\phi_{R/L}$

$$\begin{aligned}
f(x) &= -\phi_L(x)\int_0^x \phi_L(y)\phi_R(y)dy + \phi_R(x)\int_0^x \phi_L^2(y)dy \\
f'(x) &= -\phi'_L(x)\int_0^x \phi_L(y)\phi_R(y)dy - \phi_L(x)^2\phi_R(x) + \phi'_R(x)\int_0^x \phi_L^2(y)dy + \phi_R(x)\phi_L^2(x) \\
&= -\phi'_L(x)\int_0^x \phi_L(y)\phi_R(y)dy + \phi'_R(x)\int_0^x \phi_L^2(y)dy \\
f''(x) &= -\phi''_L(x)\int_0^x \phi_L(y)\phi_R(y)dy - \phi'_L(x)\phi_L(x)\phi_R(x) + \phi''_R(x)\int_0^x \phi_L^2(y)dy + \phi'_R(x)\phi_L^2(x) \\
&= (V(x) - E)\left\{-\phi_L(x)\int_0^x \phi_L(y)\phi_R(y)dy + \phi_R(x)\int_0^x \phi_L^2(y)dy\right\} - \phi_L(x)W(x) \\
&= (V(x) - E)f(x) - \phi_L(x)\phi_L(1)
\end{aligned}$$

We check the boundary conditions

$$\begin{aligned}
f(0) &= -\phi_L(0)\int_0^0 \phi_L(y)\phi_R(y)dy + \phi_R(0)\int_0^0 \phi_L^2(y)dy = 0 \\
f'(0) &= -\phi'_L(0)\int_0^0 \phi_L(y)\phi_R(y)dy + \phi'_R(x)\int_0^0 \phi_L^2(y)dy = 0
\end{aligned}$$

It is thus clear that $\frac{\partial \phi_L(x)}{\partial E}$ is the solution of the problem.

17. One just needs to take $x = 1$ in the formula, using $\phi_R(1) = 0$ and recalling the result for the Green function, we get

$$\frac{\partial \phi_L(1,E)}{\partial E} = -\int_0^1 \phi_L(y)\phi_R(y)dy = -\phi_L(1,E)\int_0^1 G_E(y,y)dy \tag{17}$$

18. In the operator form, we know that the Green function obeys $(\hat{H} - E\mathbb{I})\hat{G}_E = \mathbb{I}$ or $\hat{G}_E = \mathbb{I}/(\hat{H} - E\mathbb{I})$. Therefore, for matrices, the choice $\mathbf{A} = \mathbf{H} - E\mathbb{I}$ gives $\mathbf{A}^{-1} = \mathbf{G}_E$ and $\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}E} = -\mathbb{I}$ so that

$$\frac{1}{\det(\mathbf{H} - E\mathbb{I})}\frac{\mathrm{d}}{\mathrm{d}E}\det(\mathbf{H} - E\mathbb{I}) = -\operatorname{Tr}\mathbf{G}_E \tag{18}$$

which suggests the definition for the operator form $\hat{G}_E$.

19. We notice two things: first $\operatorname{Tr}\hat{G}_E = \int_0^1 G_E(x,x)dx = -\frac{1}{\phi_L(1,E)}\frac{\partial \phi_L(1,E)}{\partial E} = -\frac{\partial}{\partial E}\ln|\phi_L(1,E)|$. Second,

$$\frac{1}{\det(\hat{H} - E\mathbb{I})}\frac{\partial}{\partial E}\det(\hat{H} - E\mathbb{I}) = \frac{1}{\det(E\mathbb{I} - \hat{H})}\frac{\partial}{\partial E}\det(E\mathbb{I} - \hat{H}) = \frac{\mathrm{d}}{\mathrm{d}E}\ln\left|\det(\hat{H} - E\mathbb{I})\right| \tag{19}$$

Assuming positivity of the arguments of the logs, we can integrate the relation

$$\frac{\mathrm{d}}{\mathrm{d}E}\ln\det(\hat{H} - E\mathbb{I}) = \frac{\partial}{\partial E}\ln\phi_L(1,E) \tag{20}$$

from 0 to E to get the Gel'fand-Yaglom formula (more generally, the integration can actually be performed in the complex plane by assuming that E is a complex variable). N.B.: $\det(\hat{H})$ may be singular is the spectrum of $\hat{H}$ has an eigenvalue that is zero. Another way to regularize the formula is to introduce some reference Hamiltonian $\hat{H}_0$ (say the free particle) and use

$$\frac{\det(\hat{H} - E\mathbb{I})}{\det(\hat{H}_0 - E\mathbb{I})} = \frac{\phi(1,E)}{\phi_0(1,E)} \tag{21}$$