Master Physics of Complex Systems Year 2019-2020

Exam on Mathematical tools

3 hours

Wednesday January 8th

You are allowed to use only your notes and documents distributed during the lectures.
Do not mind about not doing everything, the exam is too long.
*As a guide, we give an estimate of points **[n]** for each question (the final mean of the class will be rescaled so these are indicative only).*
*We add a star * for difficult or lengthly questions.*
Many parts and questions are actually independent and can be done without solving the previous ones (but using intermediate results). Try to read and understand carefully the whole problem.
This exam uses the following topics: complex analysis, Fourier transform, saddle-point methods, Green's functions, orthogonal polynomials.

The Poisson summation formula [∼3 points]

We consider a function $g(x)$ of a real variable x, that is periodic of period 1. We recall that it can be expand in Fourier series, in the following way

$$g(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i2\pi nx} \ , \quad \text{where} \ \ c_n = \int_{-1/2}^{1/2} g(x)\, e^{-i2\pi nx} dx \tag{1}$$

1. **[1]** Give the expression of the Fourier transform $G(k)$ of $g(x)$, defined as $G(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} g(x)\, e^{-ikx} dx$, as a function of the c_n and delta functions.

2. **[1]** We consider the Dirac comb $g(x) = \sum_{n=-\infty}^{+\infty} \delta(x-n)$. What is $G(k)$?

3. **[1*]** Let $f(x)$ be a function of a real variable x and $F(k)$ its Fourier transform. Infer the **Poisson summation formula**

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x)\, e^{-i2\pi mx} dx \tag{2}$$

Some properties of Bessel functions [∼12 points]

Prime's means derivative in this exercise. Bessel functions are solutions of the equation of the unknown real function $y(x)$ of real variable x for $\nu \in \mathbb{R}$:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0 \tag{3}$$

4. **Series expansion.** We look for a solution of the form $y(x) = x^\mu \sum_{k=0}^{+\infty} a_k x^k$ with $\mu \in \mathbb{R}$, $a_0 \neq 0$.

   a) [**1**] By considering the $x \to 0$ behavior, show that $\mu = \pm\nu$.

   b) [**2***] By convention, we impose that $a_0 = 1/2^\nu \Gamma(\nu + 1)$ (*use your notes on the Gamma function*). We consider for simplicity the case $\nu \geq 0$, show (check) that

   $$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m} \tag{4}$$

   We admit that this expansion remains valid for all $\nu \in \mathbb{R}$.

5. [**2***] **Generating function.** Prove the following relation for $t \in \mathbb{C}$ (we recall that by convention $n! = \infty$ for $n = -1, -2, -3, \ldots$):

   $$G(x,t) = e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{+\infty} t^n J_n(x) \tag{5}$$

   In passing, you may show that $J_n(x) = (-1)^n J_{|n|}(x)$ *when* $n < 0$.

6. [**1.5**] **Integral representation.** First show that $e^{ix\sin\theta} = \sum_{n=-\infty}^{+\infty} e^{in\theta} J_n(x)$ and then that

   $$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta} e^{-in\theta} d\theta\ , \quad \text{and} \quad J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta} d\theta \tag{6}$$

7. [**1**] **Complex integral representation.** For any counterclockwise contour C enclosing $t = 0$, show that the following two equalities, for $n \in \mathbb{Z}$:

   $$J_n(x) = \frac{1}{2i\pi}\oint_C t^{-n-1} e^{\frac{x}{2}(t-1/t)} dt = \frac{1}{2i\pi}\left(\frac{x}{2}\right)^n \oint_C u^{-n-1} e^{u - \frac{x^2}{4u}} du \tag{7}$$

8. [**2**] **Recursion relations.** Prove the following two relations:

   $$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{8}$$
   $$2n\, J_n(x) = x(J_{n-1}(x) + J_{n+1}(x)) \tag{9}$$

9. [**2.5***] **Asymptotic behavior.** Show that, when $x \to \infty$ (in particular $x \gg |n|$)

   $$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{10}$$

Trace formula on the rectangular billiard [∼18+4 points]

We consider the problem of calculating the density of states of a free particle confined in a two-dimensional rectangular billiard with periodic boundary conditions (see Figure 1). The **density of states** is formally defined for a discrete spectrum as

$$\rho(E) = \sum_{\vec{n}} \delta(E - E_{\vec{n}}) \tag{11}$$

using Dirac peaks centered over the energies $E_{\vec{n}}$, with $\vec{n}$ the vector containing the quantum numbers.

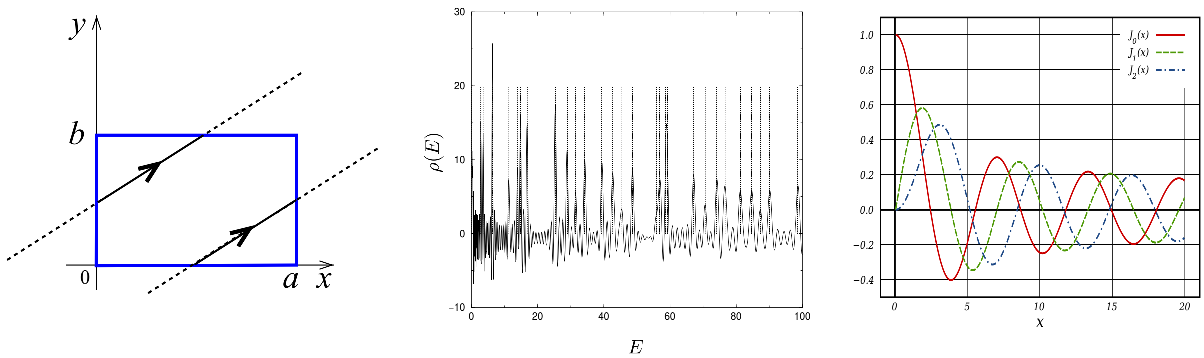

Figure 1: **Left:** Sketch of the billiard and a typical classical periodic trajectory of length $L_{1,1} = \sqrt{a^2 + b^2}$. **Middle:** Comparison between exact spectrum and its reconstruction from the trace formula truncated to the first 250 terms. **Right:** Bessel functions.

In quantum mechanics, it corresponds to the following linear differential eigenvalue problem

$$(\Delta + E_{\vec{n}})\psi_{\vec{n}}(x, y) = 0 \tag{12}$$

with $x \in [0, a]$ and $y \in [0, b]$ are the two-dimensional cartesian coordinates and the rectangle has dimensions a and b. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the laplacian operator. Periodic boundary conditions means that the solutions satisfy to $\psi(x + a, y) = \psi(x, y)$ and $\psi(x, y + b) = \psi(x, y)$. The $\{E_{\vec{n}}, \psi_{\vec{n}}(x, y)\}$ are eigenvalues and *normalized* eigenfunctions of the problem.

Using Poisson formula

10. **[2]** Using plane waves $e^{i(k_1 x + k_2 y)}$ as eigenfunctions, show that the eigenvalues (energies) read

$$E_{n_1,n_2} = \left(\frac{2\pi}{a} n_1\right)^2 + \left(\frac{2\pi}{b} n_2\right)^2 \tag{13}$$

and give the possible values of the quantum numbers n_1 and n_2. Interpret their signs.

11. **[4*]** Using twice the Poisson formula (2) introducing two variables x_1 and x_2, and switching to the following polar coordinates (r, θ) with $x_1 = \frac{ar}{2\pi} \cos\theta$ and $x_2 = \frac{br}{2\pi} \sin\theta$, show that the density of states can be rewritten as

$$\rho(E) = \frac{S}{4\pi} \sum_{m_1,m_2=-\infty}^{+\infty} J_0\left(L_{m_1,m_2}\sqrt{E}\right), \qquad \text{where} \quad L_{m_1,m_2} = \sqrt{(m_1 a)^2 + (m_2 b)^2} \tag{14}$$

with S a constant to be determined and J_0 the Bessel function of order 0 defined in (6). *The series may not be convergent but we allow ourselves to writes summations of this type without care in the following.*

12. [**2***] Geometrically speaking, what are the possible **periodic** classical trajectories of a particle is such a rectangular billiard with periodic boundary conditions? In particular, what are the possible lengths of these periodic orbits?

13. [**1**] Show that there is a non-oscillating contribution to the density of states $\bar{\rho}(E)$ and give its expression. Thus, we'll use the splitting $\rho(E) = \bar{\rho}(E) + \tilde{\rho}(E)$.

14. [**1**] We consider the semi-classical limit $L_{m_1,m_2}\sqrt{E} \gg 1$, give an explicit approximation for the oscillating part of the density of states $\tilde{\rho}(E)$ as a function of $\sqrt{E}$, L_{m_1,m_2} and S using the asymptotical behavior of Bessel functions.

Using the Green's function

In quantum mechanics, one defines a special kind of Green's function G_z, called the resolvant, as the solution of the following problem

$$(z - \hat{H})G_z(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \quad + \quad \text{boundary conditions} \tag{15}$$

where $\hat{H}$ is the Hamiltonian ($\hat{H} = -\Delta$ for the free particle) and $z \in \mathbb{C}$ a complex number.

15. **Density of states from the Green's function.**
    a) [**1**] Give the spectral representation of the operator $\hat{G}_z$ associated to $G_z(\vec{r}, \vec{r}\,')$ in terms of the $E_{\vec{n}}$ and the kets $|\psi_{\vec{n}}\rangle$.
    b) [**1**] Infer the expression of $\operatorname{Tr}\hat{G}_z$ both as a series involving the $E_{\vec{n}}$ and an integral over $\vec{r}$.
    c) [**2**] We now set $z = E + i\varepsilon$ with $\varepsilon > 0$ a small number for which one implicitly assumes $\varepsilon \to 0$. Prove the following relation involving the retarded Green's function $\hat{G}_E^+ = \lim_{\varepsilon\to 0} \hat{G}_z$

    $$\rho(E) = -\frac{1}{\pi} \operatorname{Im} \int G_E^+(\vec{r}, \vec{r}\,) d\vec{r} \tag{16}$$

16. **Free particle Green's function:** We now consider a particle in free space (*not a billiard*) with boundary conditions such that the Green's function vanishes when $\|\vec{r}\,\| \to \infty$.
    a) [**2**] Show that the corresponding solution $G_E^{0,+}$ can be expressed as

    $$G_E^{0,+}(\vec{r}, \vec{r}\,') = \iint_{\mathbb{R}^2} \frac{dk_x dk_y}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{E - \vec{k}^{\,2} + i\varepsilon} \tag{17}$$

    b) [**Bonus+4****] Using the trick $\dfrac{1}{\vec{k}^{\,2} - E - i\varepsilon} = \displaystyle\int_0^{+\infty} dt\, e^{-t(\vec{k}^{\,2} - E - i\varepsilon)}$, and the expression of the Bessel function (7), show that this integral gives the result

    $$\operatorname{Im} G_E^{0,+}(\vec{r}, \vec{r}\,') = -\frac{1}{4} J_0\left(\sqrt{E}\|\vec{r} - \vec{r}\,'\|\right) \tag{18}$$

17. **Rectangular billiard Green's function:** periodic boundary conditions now impose that $G_E^+(x + na, y + mb, x', y') = G_E^+(x, y, x', y')$ for all integer n, m.
    a) [**1**] Give two main arguments supporting the fact that the following ansatz is the Green's function for the confined particle

    $$G_E^+(x, y, x', y') = \sum_{n,m=-\infty}^{\infty} G_E^{0,+}(x + na, y + mb, x', y') \tag{19}$$

    b) [**1**] Recover the trace formula (14) from the above results.