The Poisson summation formula [∼3 points]

We consider a function $g(x)$ of a real variable x, that is periodic of period 1. We recall that it can be expand in Fourier series, in the following way

$$g(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i2\pi nx} \ , \qquad \text{where} \ \ c_n = \int_{-1/2}^{1/2} g(x)\, e^{-i2\pi nx} dx \tag{1}$$

1. **[1]** $G(k) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} c_n \int_{-\infty}^{+\infty} e^{i(2\pi n-k)x} dx = \sqrt{2\pi} \sum_{n=-\infty}^{+\infty} c_n\, \delta(k-2\pi n)$

2. **[1]** $c_n = \int_{-1/2}^{1/2} \sum\limits_{m=-\infty}^{+\infty} \delta(x-m)\, e^{-i2\pi nx} dx = \int_{-1/2}^{1/2} \delta(x)\, e^{-i2\pi nx} dx = 1$ (only $m=0$ in the interval of the integral).

   Thus, $G(k) = \sqrt{2\pi} \sum\limits_{n=-\infty}^{+\infty} \delta(k-2\pi n)$

3. **[1*]** Using Parseval-Plancherel equality : $\int_{-\infty}^{+\infty} f(x)g(x)dx = \int_{-\infty}^{+\infty} F(k)G(k)dk$, we get

$$\int_{-\infty}^{+\infty} f(x) \sum_{n=-\infty}^{+\infty} \delta(x-n)dx = \sum_n f(n) \tag{2}$$

$$= \int_{-\infty}^{+\infty} F(k)\sqrt{2\pi} \sum_{n=-\infty}^{+\infty} \delta(k-2\pi n)dk = \sqrt{2\pi} \sum_{m=-\infty}^{+\infty} F(2\pi m) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x) e^{-i2\pi mx} dx \tag{3}$$

Some properties of Bessel functions [∼12 points]

4. a) **[1]** the behavior of the ansatz when $x \to 0$ is $y(x) \simeq x^m u$. Injecting this in the differential equation gives for the $x^{\mu-2}$ coefficient $\mu(\mu-1)+\mu-\nu^2 \Rightarrow \mu = \pm\nu$.

   b) **[2*]** with $\nu \geq 0$ inserting the ansatz and setting to 0 the coefficient in front of $x^{\nu+k-2}$ gives the relation $a_k = -\frac{a_{k-2}}{k(2\nu+k)}$. Furthermore, the term $a_1(2)x^{\nu-1}$ is alone which sets $a_1(2\nu+1)=0$ so only even indices k remain. Writting the expansion $y(x) = x^\nu \sum_{m=0}^{+\infty} c_m x^{2m}$ one obtains

$$c_m = \frac{(-1)^m}{2^{2m} m!(1+\nu)\cdots(m+\nu)} a_0 \tag{4}$$

   taking $a_0 = 1/2^\nu \Gamma(\nu+1)$ and using $\Gamma(m+\nu+1) = (1+\nu)\cdots(m+\nu)\Gamma(\nu+1)$, one gets

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m} \tag{5}$$

   A similar reasoning allows one to show that the formula extends to $\nu < 0$.

5. **[2*]** Let us expand algebraically the formula using the exponential expansion and the binomial formula

$$G(x,t) = e^{\frac{x}{2}(t-1/t)} = \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\frac{x}{2}(t-t^{-1})\right)^k = \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\frac{x}{2}\right)^k \sum_{\substack{r,s=0 \\ r+s=k}}^{k} \frac{(r+s)!}{r!s!} t^r (-t)^{-s} \tag{6}$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{r+s} \frac{1}{r!s!} t^r t^{-s} \tag{7}$$

   making the change of variable $n = r-s \in \mathbb{Z}$, one formally obtains

$$G(x,t) = \sum_{n=-\infty}^{+\infty} t^n \left(\frac{x}{2}\right)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)!s!} \left(\frac{x}{2}\right)^{2s} \tag{8}$$

   in which we recognize the series expansion of $J_n(x)$ for $\nu = n$ integer. Yet, one has to be careful with negative n in this formula. Formally, some negative factorial appear whenever $s < -|n|$ when $n < 0$. The convention is that these terms are infinite so they kill the contribution. For $n < 0$, then the definition of J_n reads

$$J_n(x) = \left(\frac{x}{2}\right)^{-|n|} \sum_{s=|n|}^{\infty} \frac{(-1)^s}{(s-|n|)!s!} \left(\frac{x}{2}\right)^{2s} = \left(\frac{x}{2}\right)^{-|n|} \sum_{s'=0}^{\infty} \frac{(-1)^{s'+|n|}}{(s'+|n|)!s'!} \left(\frac{x}{2}\right)^{2s'+2|n|} = (-1)^{|n|} J_{|n|}(x) \tag{9}$$

so these are the same functions up to a sign and the identification with the series holds assuming the previous definition of J_ν holds for all $\nu \in \mathbb{R}$.

6. **[1.5] Integral representation.** One sets $t = e^{i\theta}$ to get $e^{ix\sin\theta} = \sum_{n=-\infty}^{+\infty} e^{in\theta} J_n(x)$ and then one integrates the generating function as

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta} e^{-in\theta} d\theta = \sum_m J_m(x) \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta = J_n(x) \;, \tag{10}$$

Then, for $\nu = 0$ one can make the change of variable $\theta \Rightarrow \pi/2 - \theta$ that changes the sine function into a cosine and notice that the origin of the integral domain doesn't matter since the cosine function is periodic

$$J_0(x) = \frac{1}{2\pi}\int_{\varphi_0}^{\varphi_0+2\pi} e^{ix\cos\theta} d\theta \tag{11}$$

This is a form commonly found in physics and that we used in the tutorial.

7. **[1] Complex integral representation.** Integrating the generating function over t yields

$$\frac{1}{2i\pi}\oint_C t^{-n-1} e^{\frac{x}{2}(t-1/t)} dt = \sum_m J_m(x) \frac{1}{2i\pi}\oint_C t^{m-n-1} dt = J_n(x) \tag{12}$$

This can be also be seen as the n^{th} derivative w.r.t. t at $t = 0$, following the principle of generating functions, associated to the Cauchy formula. The following representation is obtained from the change of variables $t = 2u/x$:

$$J_n(x) = \frac{1}{2i\pi}\left(\frac{x}{2}\right)^n \oint_C u^{-n-1} e^{u-\frac{x^2}{4u}} du \tag{13}$$

8. **[2] Recursion relations.** From the generating function $\frac{\partial G}{\partial x} = \sum_n t^n J_n'(x) = \frac{1}{2}(t - 1/t)G(x,t) = (\sum_n J_n t^{n+1} - \sum_n J_n t^{n-1})/2$ yields the first equality by identification of powers of t. The second one is obtained from $\frac{\partial G}{\partial t} = \sum_n t^n J_n'(x) = \frac{x}{2}(1 + 1/t^2)G(x,t) = \sum_n n t^{n-1} J_n(x)$. Finally,

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \text{ and } 2n\, J_n(x) = x(J_{n-1}(x) + J_{n+1}(x)) \tag{14}$$

9. **[2.5*] Asymptotic behavior.** With the notation of the lecture, we have $h(\theta,x) = x\sin\theta - n\theta$, $h'(\theta,x) = x\cos\theta - n$, $h''(\theta,x) = -x\sin\theta$, so the saddle points satisfy (approximations use $x \gg n$ assuming $n > 0$ for simplicity) $\cos\theta_c = n/x \simeq 0$ so $\theta_{c,\pm} = \pm\arccos(n/x) \simeq \pm\pi/2$. We have $h''_{c,\pm} = \mp\sqrt{x^2 - n^2} \simeq \mp x$ (+: maximum, -: minimum) and $h_{c,\pm} = \pm(\sqrt{x^2-n^2} - n\arccos(n/x)) \simeq \pm(x - n\pi/2)$. We can now apply the formula summing up the two contributions

$$J_n(x) \sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{\sqrt{x^2-n^2}}}\left(e^{i(\sqrt{x^2-n^2}-n\arccos(n/x))-i\pi/4} + e^{-i(\sqrt{x^2-n^2}-n\arccos(n/x))+i\pi/4}\right) \tag{15}$$

$$\sim \sqrt{\frac{2}{\pi\sqrt{x^2-n^2}}}\cos\left(\sqrt{x^2-n^2} - n\arccos(n/x) - \frac{\pi}{4}\right) \tag{16}$$

and, with the approximation $x \gg |n|$

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{17}$$

Trace formula on the rectangular billiard [$\sim$18+4 points]

$$\rho(E) = \sum_{\vec{n}} \delta(E - E_{\vec{n}}) \tag{18}$$

$$(\Delta + E_{\vec{n}})\psi_{\vec{n}}(x,y) = 0 \tag{19}$$

with $x \in [0,a]$ and $y \in [0,b]$ are the two-dimensional cartesian coordinates and the rectangle has dimensions a and b. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the laplacian operator. Periodic boundary conditions means that the solutions satisfy to $\psi(x+a,y) = \psi(x,y)$ and $\psi(x,y+b) = \psi(x,y)$. The $\{E_{\vec{n}}, \psi_{\vec{n}}(x,y)\}$ are eigenvalues and *normalized* eigenfunctions of the problem.

Using Poisson formula

10. **[2]** Periodic boundary conditions imposes $k_1 a = 2\pi n_1$ with $n_1 \in \mathbb{Z}$ and $k_2 b = 2\pi n_2$ with $n_2 \in \mathbb{Z}$, energies are simply given by $E = \vec{k}^{\,2} = k_1^2 + k_2^2$ so $E_{n_1,n_2} = \left(\frac{2\pi}{a} n_1\right)^2 + \left(\frac{2\pi}{b} n_2\right)^2$

11. **[4*]** Let do the calculation using $f(x,y) = \delta(E - \left(\frac{2\pi}{a}\right)^2 x^2 - \left(\frac{2\pi}{b}\right)^2 y^2)$

$$\rho(E) = \sum_{n_1,n_2} \delta(E - E_{n_1,n_2}) = \sum_{n_1,n_2 \in \mathbb{Z}} f(n_1,n_2) = \sum_{m_1,m_2 \in \mathbb{Z}} \iint_{-\infty}^{+\infty} f(x_1,x_2) e^{-i2\pi(m_1 x_1 + m_2 x_2)} dx_1 dx_2 \tag{20}$$

$$= \sum_{m_1,m_2 \in \mathbb{Z}} \frac{ab}{(2\pi)^2} \int_0^{+\infty} \int_0^{2\pi} \delta(E - r^2) e^{-ir(am_1 \cos\theta + m_2 b \sin\theta)} r dr d\theta \tag{21}$$

with $S = ab$ the surface of the billiard, using $\delta(E - r^2) = \frac{1}{2r}\delta(r - \sqrt{E})$ (because $r > 0$) and noticing that $r(am_1 \cos\theta + m_2 b \sin\theta) = \vec{r} \cdot \vec{m}$ is the scalar product of the two vectors $\vec{r} = (r\cos\theta, r\sin\theta)$ and $\vec{m} = (m_1 a, m_2 b)$ so it can be rewritten as $\|\vec{r}\|\|\vec{m}\| \cos\varphi$ where φ is the relative angle between the two vectors. When θ spans $[0, 2\pi]$, then φ also spans a range of 2π so that, by periodicity of the cosine, we can rewrite the integral as

$$\rho(E) = \frac{S}{4\pi} \sum_{m_1,m_2 \in \mathbb{Z}} \frac{1}{2\pi} \int_0^{2\pi} e^{-i\sqrt{E}\sqrt{(am_1)^2 + (bm_2)^2} \cos\varphi} d\varphi = \frac{S}{4\pi} \sum_{m_1,m_2=-\infty}^{+\infty} J_0\left(L_{m_1,m_2}\sqrt{E}\right) \tag{22}$$

12. **[2*]** Periodic orbits of period (n,m) can be viewed as trajectories spaning n cells along x and m along y such that the trajectory within in the (n,m) is the same as in the first cell, formally $(x + na, y + nb) \equiv (x,y)$. Consequently the length of such orbit is the distance between two identical points in these cells, that is $L_{n,m}$. The signs of n and m gives the directions along x and y of the particle.

13. **[1]** Having a look at the behavior of the Bessel function, we see that the non-oscillating part is given only by the $n = 0, m = 0$ such that $L_{0,0} = 0$ giving a independent of energy contribution. It corresponds to no periodic orbits, just points on the billiard, so unsurprisingly $\bar{\rho} = S/4\pi$. This actually corresponds to usual the result of the semi-classical calculation in statistical mechanics performed over phase-space

$$\bar{\rho} = \iint \frac{d\vec{r}\,d\vec{p}}{(2\pi)^2} \delta(E - H(\vec{r},\vec{p})) = \frac{S}{(2\pi)^2} \int_0^\infty \delta(E - p^2) 2\pi p dp = \frac{S}{4\pi}$$

with H the classical Hamiltonian (here $H = \vec{p}^{\,2}$).

14. **[1]** Using the asymptotical behavior derived previously, we get a sum over periodic orbits

$$\tilde{\rho}(E) \sim \frac{S}{\sqrt{(2\pi)^3 \sqrt{E}}} \sum_{\text{per.orb.}p} \frac{1}{\sqrt{L_p}} \cos\left(L_p \sqrt{E} - \frac{\pi}{4}\right) \tag{23}$$

Using the Green's function

15. **Density of states from the Green's function.**
    a) **[1]** We can write $\hat{G}_z = \dfrac{1}{z - \hat{H}} = \sum_n \dfrac{|\psi_n\rangle\langle\psi_n|}{z - E_n}$
    b) **[1]** The trace corresponds to an integral over $\vec{r}$: $\operatorname{Tr}\hat{G}_z = \int G_z(\vec{r},\vec{r}) d\vec{r} = \sum_n \dfrac{1}{z - E_n}$.
    c) **[2]** We have $\operatorname{Tr}\hat{G}_E^+ = \sum_n \dfrac{1}{E - E_n + i\varepsilon}$. We recall the Feynman formula as written in the lecture, in the sens of distributions: $\dfrac{1}{x - x_0 \pm i\varepsilon} = \mathrm{PP}\dfrac{1}{x - x_0} \mp i\pi\delta(x - x_0)$, which gives $\operatorname{Tr}\hat{G}_E^+ = \sum_n \mathrm{PP}\dfrac{1}{E - E_n} - i\pi \sum_n \delta(E - E_n)$ so that $\rho(E) = -\frac{1}{\pi} \operatorname{Im} \operatorname{Tr}\hat{G}_z$ and for the particle problem $\rho(E) = -\dfrac{1}{\pi} \operatorname{Im} \int G_E^+(\vec{r},\vec{r}) d\vec{r}$.

16. **Free particle Green's function:**
    a) **[2]** We move to Fourier space : the differential equation then reads $(z - \vec{k}^{\,2}) G_E^{0,+}(\vec{k}) = 1$, which using $z = E + i\varepsilon$ and performing the inverse Fourier transform gives (translational invariance)

$$G_E^{0,+}(\vec{r},\vec{r}\,') = \iint_{\mathbb{R}^2} \frac{dk_x dk_y}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{E - \vec{k}^{\,2} + i\varepsilon} \tag{24}$$

b) **[Bonus+4**]** One writes $\vec{R} = \vec{r} - \vec{r}\,' = (X, Y)$ and using Gaussian integrals

$$G_E^{0,+}(\vec{R}) = -\int_0^{+\infty} dt \iint \frac{dk_x dk_y}{(2\pi)^2}\, e^{i(k_x X + k_y Y)} e^{-t(\vec{k}^{\,2} - E - i\varepsilon)} \tag{25}$$

$$= -\frac{1}{(2\pi)^2}\int_0^{+\infty} dt\, e^{(E+i\varepsilon)t}\sqrt{\frac{2\pi}{2t}}e^{-X^2/4t}\sqrt{\frac{2\pi}{2t}}e^{-Y^2/4t} \tag{26}$$

$$= -\frac{1}{4\pi}\int_0^{+\infty} dt\, e^{(E+i\varepsilon)t}\, t^{-1}\, e^{-R^2/4t} = -\frac{1}{4\pi}\int_L du\, u^{-1}\, e^{u - zR^2/4u} \tag{27}$$

where is the lign in the complex plane from the origin and passing through $z = E + i\varepsilon$ ... (contour integral discussion) ... we arrive at

$$\operatorname{Im} G_E^{0,+}(\vec{r}, \vec{r}\,') = -\frac{1}{4} J_0\left(\sqrt{E}\|\vec{r} - \vec{r}\,'\|\right) \tag{28}$$

17. **Rectangular billiard Green's function:**

a) **[1]** i) All terms $G_E^{0,+}(x + na, y + mb, x', y')$ satisfy to the differential equation of the Green's function. Only the $(n = 0, m = 0)$ term will produce a δ peak inside the billiard.
ii) The G_E^+ function clearly satisfy to boundary conditions since applying translation to x and y amounts the shifting the sum indices that goes from $-\infty$ to $+\infty$.

b) **[1]** We now get

$$\rho(E) = -\frac{1}{\pi}\operatorname{Im}\int G_E^+(\vec{r}, \vec{r})d\vec{r} = -\frac{1}{\pi}\sum_{n,m=-\infty}^{+\infty}\operatorname{Im}\int_S G_E^{0,+}(\vec{r} + (na, mb), \vec{r})d\vec{r} \tag{29}$$

$$= \frac{1}{4\pi}\sum_{n,m=-\infty}^{+\infty}\underbrace{\int_S d\vec{r}}_{=S}\,\underbrace{J_0\left(\sqrt{E}\|(na, mb)\|\right)}_{\text{independent of } \vec{r}} = \frac{S}{4\pi}\sum_{n,m=-\infty}^{+\infty} J_0\left(L_{n,m}\sqrt{E}\right) \tag{30}$$