# Exam on Mathematical tools 

3 hours

Wednesday January 20th

You are allowed to use only your notes and documents distributed during the lectures.
Do not mind about not doing everything, the exam is too long.
As a guide, we give an estimate of points [ $\boldsymbol{n}$ ] for each question (the final mean of the class will be rescaled so these are indicative only).
Many parts and questions are actually independent and can be done without solving the previous ones (but using intermediate results). Try to read and understand carefully the whole problem.
This exam uses the following topics: variational calculus, complex analysis, asymptotic expansions, Green's functions, Gaussians, orthogonal polynomials.

## Some Fourier integral [~3 points]

We consider the function

$$
\begin{equation*}
C(\theta, \lambda)=\frac{\sinh (\lambda)}{\cosh (\lambda)+\cos \theta}, \quad \text { with } \lambda>0 \tag{1}
\end{equation*}
$$

and $\theta \in[0,2 \pi]$ an angle variable. We recall the definition of discrete Fourier transform.

$$
\begin{equation*}
C(\theta, \lambda)=\sum_{n=-\infty}^{+\infty} C_{n}(\lambda) e^{-i n \theta} \tag{2}
\end{equation*}
$$

1. [2] Compute the $C_{n}(\lambda)$ through some Fourier transform over $\theta$.
2. [1] Check your result by resumming the series explicitly.

## Expansions of the complementary error function [ $\sim 3$ points]

One defines the complementary error function as

$$
\begin{equation*}
\operatorname{Erfc}(x)=\frac{2}{\sqrt{\pi}} \int_{x}^{+\infty} e^{-t^{2}} \mathrm{~d} t \tag{3}
\end{equation*}
$$

3. [1] Find the Taylor expansion of the $\operatorname{Erfc}(x)$ function at $x=0$.
4. [2] Show that the asymptotic expansion when $x \rightarrow \infty$ of the $\operatorname{Erfc}()$ function takes the form

$$
\begin{equation*}
\operatorname{Erfc}(x) \sim L(x)\left(\sum_{n=0}^{+\infty} \frac{a_{n}}{x^{n}}\right) \tag{4}
\end{equation*}
$$

in which you have to give the explicit form of the leading term $L(x)$ and the $a_{n}$ coefficients.

## Field theories [ $\sim 5$ points]

Elastic string - We consider a problem described by a scalar field $\psi(x, t)$ which equations of motion are obtained by the minimization of the action $\mathcal{S}[\psi]$, expressed using the Lagrangian density $\mathcal{L}$ as

$$
\begin{equation*}
\mathcal{S}[\psi]=\int_{0}^{L} \int_{0}^{T} d x d t \mathcal{L}\left(\psi, \partial_{t} \psi, \partial_{x} \psi\right) \tag{5}
\end{equation*}
$$

with $L$ and $T$ some fixed parameters.
5. [1] Write down the Euler-Lagrange equation $\frac{\delta S}{\delta \psi(x, t)}=0$ in terms of partial derivatives of $\mathcal{L}$.
6. [1] We consider the Lagrangian density $\mathcal{L}=\frac{\rho}{2}\left(\left(\partial_{t} \psi\right)^{2}-c^{2}\left(\partial_{x} \psi\right)^{2}\right)$ with $\rho$ and $c$ two constants. Deduce the equation of motion for $\psi(x, t)$ ? What is the physical meaning of $c$ ?

Non-linear Schrödinger equation - We now consider a complex field $\psi(\vec{x}, t)$ describing a quantum problem with $\psi^{*}(\vec{x}, t)$ its complex conjugate. The position variable $\vec{x}$ is now in arbitrary dimension. One can take either the real and imaginary part of $\psi$ as independent variables or, equivalently, use $\psi$ and $\psi^{*}$ as independent variables. Thus, the action and Lagrangian density take the form

$$
\begin{equation*}
\mathcal{S}\left[\psi, \psi^{*}\right]=\iint_{0}^{T} d \vec{x} d t \mathcal{L}\left(\psi, \psi^{*}, \partial_{t} \psi, \partial_{t} \psi^{*}, \vec{\nabla} \psi, \vec{\nabla} \psi^{*}\right), \quad \text { with } \quad(\vec{\nabla})_{j}=\partial_{x_{j}} \tag{6}
\end{equation*}
$$

7. [1] Write down the Euler-Lagrange equations for the field $\psi(\vec{x}, t)$.
8. [2] We consider the following Lagrangian density with $V(x)$ a potential and $g$ a constant:

$$
\begin{equation*}
\mathcal{L}=i \frac{\hbar}{2}\left(\psi^{*} \partial_{t} \psi-\psi \partial_{t} \psi^{*}\right)-\frac{\hbar^{2}}{2 m} \vec{\nabla} \psi \cdot \vec{\nabla} \psi^{*}-V(x)|\psi|^{2}-g|\psi|^{4} \tag{7}
\end{equation*}
$$

What are the equations of motion for $\psi(\vec{x}, t)$ ? How are they related?

## Green's function for the damped harmonic oscillator [ $\sim 14$ points]

Reconsider the driven damped harmonic oscillator with the same notations as in the lecture (mass=1):

$$
\begin{equation*}
\left[\frac{\mathrm{d}^{2}}{\mathrm{~d} t^{2}}+2 \gamma \frac{\mathrm{~d}}{\mathrm{~d} t}+\omega_{0}^{2}\right] x(t)=f(t) \tag{8}
\end{equation*}
$$

with $\gamma>0$ the damping parameter, $\omega_{0}>0$ the free pulsation and $f(t)$ an arbitrary forcing. We are looking causal solutions to the problem. You will use the notations $\tau=t-t^{\prime}$ and $\Gamma=\sqrt{\gamma^{2}-\omega_{0}^{2}}$.
9. [0.5] Recall which equation and which conditions are satisfied by the Green's function $G\left(t, t^{\prime}\right)$ associated to (8).
10. [1] Recall the generic solutions of the homogeneous equation associated to (8). Second, recall the particular solution of (8), simply denoted by $x(t)$, as a function of $f(t)$ and $G\left(t, t^{\prime}\right)$.
11. The overdamped limit $\gamma>\boldsymbol{\omega}_{\mathbf{0}}$. We take $f(t)$ to be a white noise, characterizing a thermal equilibrium at temperature $T_{\text {eq }}$, ie. $\langle f(t)\rangle=0$ and $\left\langle f(t) f\left(t^{\prime}\right)\right\rangle=A \delta\left(t-t^{\prime}\right)$ with $A$ a constant and $\langle\cdots\rangle$ the statistical average. We define the time correlator by $C(T)=\langle x(t+T) x(t)\rangle$.
a) [2] Show carefully that one has

$$
\begin{equation*}
G\left(t, t^{\prime}\right)=\Theta(\tau) \frac{e^{-\gamma \tau}}{\Gamma} \sinh (\Gamma \tau) \tag{9}
\end{equation*}
$$

b) [1] Express $C(T)$ under an integral form involving $G$.
c) [2] Compute explicitly $C(T)$.
d) [1] By considering $C(0)=\left\langle x^{2}(t)\right\rangle$ and using a famous and simple theorem from statistical mechanics, give the expression of $A$ as a function of $\gamma$ and $k_{B} T_{\text {eq }}$.
e) $[2]$ Compute the diffusion coefficient $D=\left.\frac{\mathrm{d}}{\mathrm{d} T}\left\langle[x(t+T)-x(t)]^{2}\right\rangle\right|_{T=0}$.
12. Critical case $\gamma=\boldsymbol{\omega}_{\mathbf{0}}$.
a) [2] Find the explicit form of the Green's function expressed as a function of $\gamma$ using Fourier transform and complex analysis calculation.
b) [2] Prove again the result but using solutions from the homogeneous equation and the usual boundary conditions satisfied by the Green's function.
c) $[\mathbf{0 . 5}]$ Recover the result through a proper limit of the result (9).

## Hermite's polynomials [ $\sim 24$ points]

We recall the main results on Hermite's polynomials $H_{n}(x)$ using the notations of the lecture. We recall the table of the lecture notes but we did not prove all formulas.

| Differential equation | $y^{\prime \prime}(x)-2 x y^{\prime}(x)+2 n y(x)=0$ |
| :--- | :---: |
| Rodrigues formula: $w(x)=e^{-x^{2}}$ | $H_{n}(x)=(-1)^{n} e^{x^{2}} \frac{\mathrm{~d}^{n}}{\mathrm{~d} x^{n}} e^{-x^{2}}$ |
| Parameters | $S=]-\infty, \infty\left[, \quad \lambda_{n}=2 n, \quad c_{n}=(-1)^{n}, \quad N_{n}=2^{n} n!\sqrt{\pi}\right.$ |
| Generating function | $G(x, t)=e^{-t^{2}+2 t x}=\sum_{n=0}^{\infty} H_{n}(x) \frac{t^{n}}{n!}$ |
| Recurrence relation | $H_{n+1}(x)=2 x H_{n}(x)-2 n H_{n-1}(x)$ |

## General properties

13. [1] Explain through a calculation why the weight function $w(x)$ is a Gaussian.
14. [2] Starting only from the knowledge of the differential equation and the recurrence relation, show that one has (Hint: consider injecting $y=H_{n+1}-2 x H_{n}+2 n H_{n-1}$ in the differential equation.)

$$
\begin{equation*}
H_{n}^{\prime}(x)=2 n H_{n-1}(x) \tag{10}
\end{equation*}
$$

15. Generating function. We use the definition $G(x, t)=\sum_{n=0}^{\infty} H_{n}(x) \frac{t^{n}}{n!}$ and we want to compute $G(x, t)$ using Rodrigues formula only.
a) [1] Schlaefli representation. Explain why one has

$$
\begin{equation*}
H_{n}(x)=(-1)^{n} \frac{n!}{2 i \pi} e^{x^{2}} \oint_{\mathcal{C}} \frac{e^{-z^{2}}}{(z-x)^{n+1}} \mathrm{~d} z \tag{11}
\end{equation*}
$$

with $\mathcal{C}$ a contour in the complex plane to be precised.
b) [2] Deduce that $G(x, t)=e^{-t^{2}+2 t x}$.
16. [2] Compute the values $H_{n}(0)$. Find, in a simple way, the explicit expression of $H_{n}(x)$ for $n=0,1,2,3$.
17. [0.5] Find the parity of $H_{n}(x)$.
18. [2] Series expansion. Starting from the expansion of the generating function, show that Hermite's polynomials can be expanded as

$$
\begin{equation*}
H_{n}(x)=\sum_{s=0}^{S_{n}} h_{s}^{n} x^{n-2 s} \tag{12}
\end{equation*}
$$

in which you have to give the expression of $S_{n}$ as a function of $n$ and the $h_{s}^{n}$ coefficients as a function of $s$ and $n$.

## Norm and orthogonality

19. [1] Write down the explicit orthogonality relation of the $H_{n}$ polynomials using the $N_{n}$ coefficient.
20. [1] Compute the norm $N_{n}$ with the help of Rodrigues formula.
21. [1] Compute explicitly the integral

$$
\begin{equation*}
F(s, t)=\int_{-\infty}^{\infty} e^{-x^{2}} G(x, s) G(x, t) \mathrm{d} x \tag{13}
\end{equation*}
$$

22. [1.5] By considering the double expansion of $F(s, t)$ as a function of both $s$ and $t$, show that the $H_{n}(x)$ polynomials form an orthogonal basis. Infer again the $N_{n}$ coefficient from this reasoning.
23. Application: Let $f(x)$ be a function that we expand over the $H_{n}(x)$ basis as $f(x)=\sum_{n=0}^{\infty} a_{n} H_{n}(x)$.
a) $[\mathbf{0 . 5}]$ Give the explicit formula allowing one to compute the $a_{n}$.
b) [1.5] Compute the $a_{n}$ in the case where $f(x)=x^{2 r}$ with $r$ integer.

## Mehler's formula and thermal density matrix

We consider the following Hamiltonian operator (harmonic oscillator):

$$
\begin{equation*}
\hat{H}=-\frac{1}{2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} x^{2}}+\frac{1}{2} x^{2}, \quad \varphi_{n}(x)=H_{n}(x) \frac{e^{-x^{2} / 2}}{\sqrt{N_{n}}} \tag{14}
\end{equation*}
$$

24. [1] Show that the $\varphi_{n}(x)$ are normalized eigenfunctions for $\hat{H}$ and give their eigenvalues $E_{n}$.

The thermal density matrix $\rho(x, y, \beta)$ describes the quantum statistical features of $\hat{H}$. It satisfies to the following diffusion equation

$$
\begin{equation*}
\frac{\partial \rho}{\partial \beta}=-\hat{H}_{x} \rho \tag{15}
\end{equation*}
$$

in which $\hat{H}_{x}$ means that $\hat{H}_{x}$ acts on the $x$-variable only. It is then natural to look for a solution that is an expansion over the $\varphi_{n}$ basis:

$$
\begin{equation*}
\rho(x, y, \beta)=\sum_{n=0}^{\infty} c_{n}(\beta) \varphi_{n}(x) \varphi_{n}(y) \tag{16}
\end{equation*}
$$

25. [1] Show that $c_{n}(\beta)=e^{-\beta / 2} e^{-\beta n}$.

For $0 \leq t<1$, Mehler's formula reads

$$
\begin{align*}
\sum_{n=0}^{\infty} t^{n} \varphi_{n}(x) \varphi_{n}(y) & =\frac{1}{\sqrt{\pi\left(1-t^{2}\right)}} \exp \left(\frac{t}{1-t^{2}} 2 x y-\frac{1+t^{2}}{1-t^{2}} \frac{x^{2}+y^{2}}{2}\right)  \tag{17}\\
& =\frac{1}{\sqrt{\pi\left(1-t^{2}\right)}} \exp \left(\frac{x^{2}-y^{2}}{2}-\frac{(x-y t)^{2}}{1-t^{2}}\right) \tag{18}
\end{align*}
$$

26. [1] After explaining the Gaussian equality $e^{-x^{2}}=\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^{2}+2 i x u} \mathrm{~d} u$, show that

$$
\begin{equation*}
H_{n}(x)=\frac{(-2 i)^{n}}{\sqrt{\pi}} e^{x^{2}} \int_{-\infty}^{\infty} u^{n} e^{-u^{2}+2 i x u} \mathrm{~d} u \tag{19}
\end{equation*}
$$

27. [2] Using (19), prove (18) through an explicit resummation of the left-hand side expansion.
28. [2] Finally, show that the thermal density matrix can be put into the compact form

$$
\begin{equation*}
\rho(x, y, \beta)=C(\beta) \exp \left(-A_{+}(\beta) \frac{(x+y)^{2}}{4}-A_{-}(\beta) \frac{(x-y)^{2}}{4}\right) \tag{20}
\end{equation*}
$$

in which $C$ and the $A_{ \pm}$are simple functions of $\beta$ to be determined explicitly.

## The end.

