Some Fourier integral [$\sim$ 3 points]

1. **[2]** $C_n(\lambda) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{\sinh\lambda}{\cosh\lambda + \cos\theta} e^{in\theta} = \frac{2\sinh\lambda}{2i\pi} \oint_{|z|=1} \mathrm{d}z \frac{z^n}{z^2 + 2\cosh(\lambda)z + 1}$ by the usual $z = e^{i\theta}$ change of variable over the unit circle. We consider $n \geq 0$ not to put the power in the and for negative n, one can check that one has $C_{-n}(\lambda) = C_n(\lambda)$. One finds two poles $z_\pm = -\cosh\lambda \pm \sinh\lambda = -e^{\mp\lambda}$. Since $\lambda > 0$, only z_+ is in the circle and applying the residue theorem and generalizing to $n < 0$ in the last equation, one gets

$$C_n(\lambda) = 2\sinh\lambda \frac{z_+^n}{z_+ - z_-} = (-1)^n e^{-\lambda|n|} \tag{1}$$

2. **[1]** Resumming the series explicitly: setting $a = e^{i\theta - \lambda}$ of module < 1:

$$C(\theta,\lambda) = \sum_{n=-\infty}^{+\infty} (-1)^n e^{-\lambda|n|} e^{in\theta} = -1 + \sum_{n=0}^{\infty} (-1)^n (a^n + (a^*)^n) = -1 + \frac{1}{1+a} + \frac{1}{1+a^*} = \frac{1 - |a|^2}{1 + a + a^* + |a|} \tag{2}$$

which gives back the compact form of $C(\theta,\lambda)$ using $|a|^2 = e^{-2\lambda}$ and $a + a^* = 2\cos(\theta)e^{-\lambda}$.

Expansions of the complementary error function [$\sim$ 3 points]

3. **[1]** One can expand $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$ but integrating the powers at ∞ is dangerous so one does the following

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \left(\int_0^{+\infty} e^{-t^2} \mathrm{d}t - \int_0^x e^{-t^2} \mathrm{d}t \right) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n} \mathrm{d}t = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} \tag{3}$$

4. **[2]** We are going to use the integration by parts trick since the x variable appears in the bounds of the integral. Then, we have a gaussian to integrate, which satisfies $\frac{\mathrm{d}}{\mathrm{d}t} e^{-t^2} = -2te^{-t^2}$, rewritten as $e^{-t^2} = -\frac{1}{2t}\frac{\mathrm{d}}{\mathrm{d}t} e^{-t^2}$ which will be then easy to integrate. Thus, for the first term, we consider

$$\int_x^{+\infty} e^{-t^2} \mathrm{d}t = \int_x^{+\infty} -\frac{1}{2t}\frac{\mathrm{d}}{\mathrm{d}t} e^{-t^2} \mathrm{d}t = \left[-\frac{1}{2t} e^{-t^2} \right|_x^{\infty} - \frac{1}{2} \int_x^{+\infty} \frac{e^{-t^2}}{t^2} \mathrm{d}t = \frac{e^{-x^2}}{2x} - \frac{1}{2} \int_x^{+\infty} \frac{e^{-t^2}}{t^2} \mathrm{d}t \tag{4}$$

Rq: another possible strategy is to set $u = t^2$ first and then to do integration by parts over $1/\sqrt{u}^n$ and e^{-u} terms which then is very close to the exponential integral example of the course. There clearly is a recursion mechanism appearing, that one generalizes to

$$I_p = \int_x^{+\infty} \frac{e^{-t^2}}{t^p} \mathrm{d}t = \int_x^{+\infty} -\frac{1}{2t^{p+1}}\frac{\mathrm{d}}{\mathrm{d}t} e^{-t^2} \mathrm{d}t = \left[-\frac{1}{2t^{p+1}} e^{-t^2} \right|_x^{\infty} - \frac{p+1}{2} \int_x^{+\infty} \frac{e^{-t^2}}{t^{p+2}} \mathrm{d}t = \frac{e^{-x^2}}{2x^{p+1}} - \frac{p+1}{2} \int_x^{+\infty} \frac{e^{-t^2}}{t^{p+2}} \mathrm{d}t$$

So $I_p = \frac{e^{-x^2}}{2x^{p+1}} - \frac{p+1}{2} I_{p+2}$, so that finally $p = 2n$ and, using $(2n-1)!! = (2n-1)(2n-3)\cdots 3.1$:

$$\mathrm{Erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \left(\sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n} \frac{1}{x^{2n}} \right) \tag{5}$$

One can check that the rest term does satisfy the convergence criteria of asymptotic series.

Field theories [$\sim$ 5 points]

Elastic string

5. **[1]** $\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial_x \psi)} = 0$

6. **[1]** $\rho \left(-\frac{\partial}{\partial t} \partial_t \psi + c^2 \frac{\partial}{\partial x} \partial_x \psi \right) = 0$ so $\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0$, d'Alembert equation with velocity c.

Non-linear Schrödinger equation

7. **[1]** There are two Euler-Lagrange equations $\frac{\delta \mathcal{S}}{\delta \psi} = 0$ and $\frac{\delta \mathcal{S}}{\delta \psi^*} = 0$. One has, following the lecture,

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} - \sum_j \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial(\partial_{x_j}\psi)} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \psi^*)} - \sum_j \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial(\partial_{x_j}\psi^*)} = 0 \tag{6}$$

8. **[2]** we get, using $\vec{\nabla}\psi \cdot \vec{\nabla}\psi^* = \sum_j (\partial_{x_j}\psi)(\partial_{x_j}\psi^*)$ and $|\psi|^2 = \psi\,\psi^*$

$$i\frac{\hbar}{2}\left(-\partial_t \psi^* - \frac{\partial}{\partial t}\psi^*\right) + \frac{\hbar^2}{2m}\sum_j \frac{\partial}{\partial x_j}(\partial_{x_j}\psi^*) - V(x)\psi^* - 2g\psi(\psi^*)^2 = 0 \tag{7}$$

$$i\frac{\hbar}{2}\left(\partial_t \psi + \frac{\partial}{\partial t}\psi\right) + \frac{\hbar^2}{2m}\sum_j \frac{\partial}{\partial x_j}(\partial_{x_j}\psi) - V(x)\psi - 2g\psi^*\psi^2 = 0 \tag{8}$$

Both equations are actually related by complex conjugation and gives the non-linear Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi + 2g|\psi|^2\psi \tag{9}$$

Green's function for the damped harmonic oscillator [$\sim$ 14.5 points]

9. **[1]** The (retarded) Green's function satisfies to

$$\hat{L}_t G = \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right] G(t,t') = \delta(t-t') \tag{10}$$

it depends on τ only and meets the causality condition $G(\tau) = 0$ for $\tau < 0$. At $t = t'$, one has two conditions : continuity of the function $G(\tau = 0^+) = G(\tau = 0^-)$ and for the derivative, by integrating the equation, one has $G'(\tau = 0^+) - G'(\tau = 0^-) = 1$.

10. **[1]** The solutions of the homogeneous equation $\hat{L}_t x = 0$ are of the form (with coefficients A, B to be determined by initial conditions):

$$x(t) = \begin{cases} [A\cos(\Omega t) + B\sin(\Omega t)]e^{-\gamma t} & \text{if} \quad \omega_0 > \gamma \\ [A + Bt]e^{-\gamma t} & \text{if} \quad \omega_0 = \gamma \\ [A\cosh(\Gamma t) + B\sinh(\Gamma t)]e^{-\gamma t} & \text{if} \quad \omega_0 < \gamma \end{cases} \tag{11}$$

where, $\Omega = \sqrt{\omega_0^2 - \gamma^2}$, or alternative forms using the $e^{\pm i\Omega t}$ and $e^{\pm \Gamma t}$ function basis. The particular solution is derived from the Green's function

$$x(t) = \int_{-\infty}^{+\infty} G(t,t')f(t')\mathrm{d}t' \; . \tag{12}$$

11. **The overdamped limit $\gamma > \omega_0$.**

    a) **[2]** We use Fourier transform following the lecture's notations $G(\omega) = \int_{-\infty}^{+\infty} G(\tau)e^{i\omega\tau}\mathrm{d}\tau$, one gets as in the lecture

$$G(\tau) = -\int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega\tau}}{\omega^2 + 2i\gamma\omega - \omega_0^2} \tag{13}$$

Here for $\gamma > \omega_0$, the two simple poles $z_\pm = -i\gamma \pm i\sqrt{\gamma^2 - \omega_0^2} = i(-\gamma \pm \Gamma)$ are purely imaginary and in the lower half of the complex plane since $\Gamma < \gamma$. Applying the residue theorem for $\tau < 0$ gives 0 and for $\tau > 0$, it gives

$$G(\tau) = -\frac{-2i\pi}{2\pi}\left(\frac{e^{-iz_+\tau}}{z_+ - z_-} + \frac{e^{-iz_-\tau}}{z_- - z_+}\right) \tag{14}$$

one finally gets

$$G(\tau) = \Theta(\tau)\frac{e^{-\gamma\tau}}{\Gamma}\sinh(\Gamma\tau) \tag{15}$$

b) **[1]** We substitute twice the particular solution in the average

$$C(T) = \langle x(t+T)x(t)\rangle = \left\langle \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}t_1'\mathrm{d}t_2' G(t+T,t_1')G(t,t_2')f(t_1')f(t_2')\right\rangle \tag{16}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}t_1'\mathrm{d}t_2' G(t+T,t_1')G(t,t_2')\left\langle f(t_1')f(t_2')\right\rangle = A\int_{-\infty}^{+\infty}\mathrm{d}t' G(t+T,t')G(t,t') \tag{17}$$

$$= A\int_{-\infty}^{+\infty}\mathrm{d}\tau G(\tau+T)G(\tau) \tag{18}$$

c) **[2]** We compute $C(T)$, assuming $T>0$ for simplicity

$$C(T) = \frac{A}{\Gamma^2}\int_{-\infty}^{+\infty}\mathrm{d}\tau\ \Theta(\tau+T)\Theta(\tau)e^{-\gamma(2\tau+T)}\sinh(\Gamma(\tau+T))\sinh(\Gamma\tau) \tag{19}$$

$$= \frac{A}{\Gamma^2}\int_{0}^{+\infty}\mathrm{d}\tau\ e^{-\gamma(2\tau+T)}\sinh(\Gamma(\tau+T))\sinh(\Gamma\tau) \tag{20}$$

$$= \frac{Ae^{-\gamma T}}{4\Gamma^2}\int_{0}^{+\infty}\mathrm{d}\tau\ e^{-2\gamma\tau}\left(e^{\Gamma(\tau+T)}-e^{-\Gamma(\tau+T)}\right)\left(e^{\Gamma\tau}-e^{-\Gamma\tau}\right) \tag{21}$$

$$= \frac{Ae^{-\gamma T}}{4\Gamma^2}\int_{0}^{+\infty}\mathrm{d}\tau\ \left(e^{\Gamma T}e^{-2(\gamma-\Gamma)\tau}+e^{-\Gamma T}e^{-2(\Gamma+\gamma)\tau}-e^{-\Gamma T}e^{-2\gamma\tau}-e^{\Gamma T}e^{-2\gamma\tau}\right) \tag{22}$$

$$= \frac{Ae^{-\gamma T}}{4\Gamma^2}\left(\frac{e^{\Gamma T}}{2(\gamma-\Gamma)}+\frac{e^{-\Gamma T}}{2(\gamma+\Gamma)}-\frac{e^{-\Gamma T}}{2\gamma}-\frac{e^{\Gamma T}}{2\gamma}\right) \tag{23}$$

$$= \frac{Ae^{-\gamma T}}{4\Gamma^2}\left(\frac{(\gamma+\Gamma)e^{\Gamma T}+(\gamma-\Gamma)e^{-\Gamma T}}{2(\gamma^2-\Gamma^2)}-\frac{1}{\gamma}\cosh(\Gamma T)\right) \tag{24}$$

$$= \frac{Ae^{-\gamma T}}{4\Gamma^2}\left(\frac{1}{\omega_0^2}\left(\gamma\cosh(\Gamma T)+\Gamma\sinh(\Gamma T)\right)-\frac{1}{\gamma}\cosh(\Gamma T)\right) \tag{25}$$

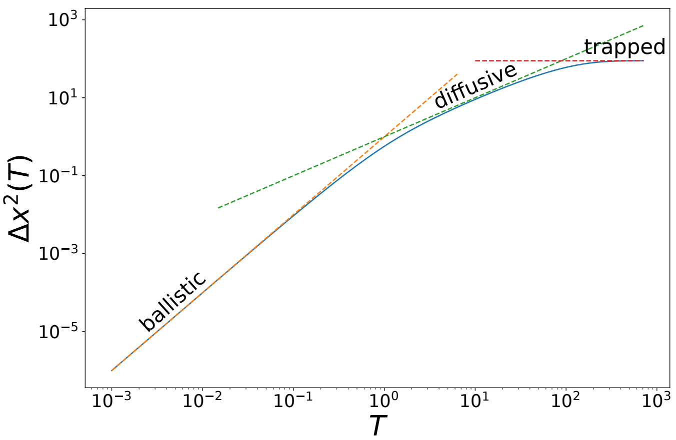
$$= \frac{Ae^{-\gamma T}}{4\omega_0^2}\left(\frac{1}{\Gamma}\sinh(\Gamma T)+\frac{1}{\gamma}\cosh(\Gamma T)\right) \tag{26}$$

d) **[1]** On one hand, equipartition theorem gives $\frac{1}{2}\omega_0^2\left\langle x^2(t)\right\rangle = \frac{1}{2}k_BT_{\mathrm{eq}}$. On the other hand, we have $C(0) = \left\langle x^2(t)\right\rangle = \frac{A}{4\gamma\omega_0^2}$. One infers that the white noise amplitude matches $A = 4\gamma k_BT_{\mathrm{eq}}$.

e) **[2]** First, we have the mean-square displacement $\Delta x^2(T) = \left\langle[x(t+T)-x(t)]^2\right\rangle = \left\langle x^2(t+T)\right\rangle + \left\langle x^2(t)\right\rangle - 2\left\langle x(t+T)x(t)\right\rangle = 2(C(0)-C(T))$:

$$\Delta x^2(T) = 2\frac{k_BT_{\mathrm{eq}}}{\omega_0^2}\left[1-\gamma e^{-\gamma T}\left(\frac{1}{\Gamma}\sinh(\Gamma T)+\frac{1}{\gamma}\cosh(\Gamma T)\right)\right] \tag{27}$$

It turns out that the short time $T\to 0$ limit is in T^2, that gives a diffusion coefficient $D=0$, because of inertia that makes the motion ballistic at short time times. At long times, the displacement is bound due to the harmonic confinement. If γ is sufficiently greater than ω_0, there exist an intermediate diffusive regime (there are two characteristic times $1/(\gamma+\Gamma)$ and $1/(\gamma-\Gamma)$) as sketched below in log log plot



12. **Critical case $\gamma = \omega_0$.**

a) **[2]** We use again Fourier transform $G(\omega) = \int_{-\infty}^{+\infty} G(\tau)e^{i\omega\tau}\mathrm{d}\tau$, one gets as in the lecture

$$G(\tau) = -\int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega^2 + 2i\gamma\omega - \gamma^2} = -\int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{e^{-i\omega\tau}}{(\omega + i\gamma)^2} \tag{28}$$

There is second order pole $z_\pm = -i\gamma$ that is purely imaginary and in the lower half of the complex plane. Applying the residue theorem for $\tau < 0$ gives 0 and for $\tau > 0$, it gives, using the formula for higher order residues :

$$G(\tau) = -\frac{-2i\pi}{2\pi} \lim_{z \to -i\gamma} \frac{\mathrm{d}}{\mathrm{d}z} \left[(z + i\gamma)^2 \frac{e^{-iz\tau}}{(z + i\gamma)^2} \right] = i(-i\tau)e^{-i(-i\gamma)\tau} = \tau\, e^{-\gamma\tau} \tag{29}$$

one finally gets

$$G(\tau) = \Theta(\tau)\tau\, e^{-\gamma\tau} \tag{30}$$

b) **[2]** The homogeneous solution for $\tau > 0$ leads to a form $G(\tau) = (A + B\tau)e^{-\gamma\tau}$ for the Green's function and we know by causality that $G(\tau) = 0$ for $\tau < 0$. Using the continuity equation $G(0^+) = 0$ gives $A = 0$. Using $G(0^+) - G(0^-) = 1$ gives $B = 1$ so we recover $G(\tau) = \Theta(\tau)\tau\, e^{-\gamma\tau}$.

c) **[0.5]** When $\gamma \to \omega_0$, setting $\Gamma \to 0$ in (15) for all t and using $\sinh(\Gamma\tau) \simeq \Gamma\tau$ gives back the result.

Hermite's polynomials [$\sim$ 24 points]

We recall the main results on Hermite's polynomials $H_n(x)$ using the notations of the lecture. We recall the table of the lecture notes but we did not prove all formulas.

Differential equation	$y''(x) - 2xy'(x) + 2ny(x) = 0$
Rodrigues formula: $w(x) = e^{-x^2}$	$H_n(x) = (-1)^n e^{x^2} \dfrac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$
Parameters	$S =]-\infty, \infty[\,,\quad \lambda_n = 2n\,,\quad c_n = (-1)^n\,,\quad N_n = 2^n n! \sqrt{\pi}$
Generating function	$G(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \dfrac{t^n}{n!}$
Recurrence relation	$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$

General properties

13. **[1]** Applying the result (6.5) of the lecture, using $p(x) = 1$ and $q(x) = -2x$ we get:

$$w(x) = \frac{1}{p(x)} \exp\left\{ \int^x \frac{q(x')}{p(x')} \mathrm{d}x' \right\} = \exp\left\{ -2 \int^x x' \mathrm{d}x' \right\} = e^{-x^2} \tag{31}$$

14. **[2]** We write $y = H_{n+1} - 2xH_n + 2nH_{n-1} = 0!$ so that $y'' - 2xy' + 2ny = 0$. We also have $H_n'' - 2xH_n' + 2nH_n = 0$, $H_{n+1}'' - 2xH_{n+1}' + 2(n+1)H_{n+1} = 0$ and $H_{n-1}'' - 2xH_{n-1}' + 2(n-1)H_{n-1} = 0$. Thus

$$\begin{aligned}
&(H_{n+1} - 2xH_n + 2nH_{n-1})'' - 2x(H_{n+1} - 2xH_n + 2nH_{n-1})' + 2n(H_{n+1} - 2xH_n + 2nH_{n-1}) = 0 \\
&(H_{n+1}'' - 2xH_n'' + 2nH_{n-1}'' - 4H_n') - 2x(H_{n+1}' - 2xH_n' + 2nH_{n-1}' - 2H_n) \\
&\qquad + 2(n+1-1)H_{n+1} - (2n)2xH_n + (2n)2(n-1+1)H_{n-1}) = 0 \\
&(H_{n+1}'' - 2xH_{n+1}' + 2(n+1)H_{n+1}) - 2H_{n+1} - 2x(H_n'' - 2xH_n' + 2nH_n) - 4H_n' + 4xH_n \\
&\qquad + 2n(H_{n-1}'' - 2xH_{n-1}' + 2(n-1)H_{n-1}) + 4nH_{n-1} = 0 \\
&-2H_{n+1} - 4H_n' + 4xH_n + 4nH_{n-1} = 0 \\
&-(2xH_n - 2nH_{n-1}) - 2H_n' + 2xH_n + 2nH_{n-1} = 0
\end{aligned}$$

so finally $H_n'(x) = 2nH_{n-1}(x)$.

15. a) **[1] Schlaefli representation**. We start from the Cauchy integral formula with $f(z) = e^{-z^2}$ that is holomorphic and taking $z_0 = x$:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z-x)^{n+1}} dz \tag{32}$$

with $\mathcal{C}$ centered around x. Combining it with $H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$, one has

$$H_n(x) = (-1)^n \frac{n!}{2i\pi} e^{x^2} \oint_{\mathcal{C}} \frac{e^{-z^2}}{(z-x)^{n+1}} \mathrm{d}z \tag{33}$$

b) **[2]** We inject this in the definition of the generating function, the countour is chosen such that $|t/(z-x)| < 1$ to ensure convergence of the sum and to contain the $x - t$ pole:

$$G(x,t) = \frac{e^{x^2}}{2i\pi} \sum_{n=0}^{\infty} t^n (-1)^n \oint_{\mathcal{C}} \frac{e^{-z^2}}{(z-x)^{n+1}} \mathrm{d}z = \frac{e^{x^2}}{2i\pi} \oint_{\mathcal{C}} \sum_{n=0}^{\infty} \left(\frac{-t}{z-x} \right)^n \frac{e^{-z^2}}{(z-x)} \mathrm{d}z \tag{34}$$

$$= \frac{e^{x^2}}{2i\pi} \oint_{\mathcal{C}} \frac{1}{1+\frac{t}{z-x}} \frac{e^{-z^2}}{(z-x)} \mathrm{d}z = \frac{e^{x^2}}{2i\pi} \oint_{\mathcal{C}} \frac{e^{-z^2}}{z-(x-t)} \mathrm{d}z = e^{x^2} e^{-(x-t)^2} = e^{x^2} e^{-x^2-t^2+2tx} \tag{35}$$

$$= e^{-t^2+2tx} \tag{36}$$

16. **[2]** The idea here is to use the generating function from the previous question when $x = 0$:

$$G(t, x=0) = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} = \sum_{k=0}^{\infty} H_k(0) \frac{t^k}{k!} \tag{37}$$

By identification, one obtains $H_{2n+1}(0) = 0$ for odd k and $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$.

Using recursion relation or integrating $H_n'(x) = 2nH_{n-1}(x)$, one finds

$$H_0 = 1 \quad H_1 = 2x \quad H_2 = 4x^2 - 2 \quad H_3 = 8x^3 - 12x \tag{38}$$

17. **[0.5]** From Rodrigues formula or the generating function, we see that $H_n(-x) = (-1)^n H_n(x)$.

18. **[2] Series expansion.** Starting from the expansion of the generating function, show that Hermite's polynomials can be expanded as

$$H_n(x) = \sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s \frac{n!}{s!(n-2s)!} (2x)^{n-2s} \tag{39}$$

in which you have to give the expression of S_n as a function of n and the h_s^n coefficients as a function of s and n.

Norm and orthogonality

19. **[1]** The scalar product requires to use the weight function $w(x)$, then we must write

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \mathrm{d}x = N_n \delta_{nm} \tag{40}$$

with N_n the norm.

20. **[1]** we write

$$N_n = \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_n(x) \mathrm{d}x = (-1)^n \int_{-\infty}^{\infty} H_n(x) \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2} \frac{\mathrm{d}^n H_n}{\mathrm{d}x^n} \mathrm{d}x \tag{41}$$

where we have used n times the integration by parts. This H_n is a polynomial of order n, $\frac{\mathrm{d}^n H_n}{\mathrm{d}x^n}$ is just the constant $n! a_n$ steming from the leading term $a_n x^n$ that can be computed from Rodrigues formula. From $\frac{\mathrm{d}}{\mathrm{d}x} e^{-x^2} = -2x e^{-x^2}$, we see that $a_n = 2^n$ (the $(-1)^n$ factor cancels with the one from the definition. Thus,

$$N_n = n! 2^n \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = n!\, 2^n \sqrt{\pi} \tag{42}$$

21. [**1**] Compute explicitly the integral

$$F(s,t) = \int_{-\infty}^{\infty} e^{-x^2} G(x,s)G(x,t)\mathrm{d}x = \sqrt{\pi}e^{2st} \tag{43}$$

22. [**1.5**] By considering the double expansion of $F(s,t)$ as a function of both s and t, show that the $H_n(x)$ polynomials form an orthogonal basis. Infer again the N_n coefficient from this reasoning.

23. **Application:** Let $f(x)$ be a function that we expand over the $H_n(x)$ basis as $f(x) = \sum_{n=0}^{\infty} a_n H_n(x)$.

    a) [**0.5**] Give the explicit formula allowing one to compute the a_n.

    b) [**1.5**] Compute the a_n in the case where $f(x) = x^{2r}$ with r integer.

Mehler's formula and thermal density matrix

We consider the following Hamiltonian operator (harmonic oscillator):

$$\hat{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 \ , \quad \varphi_n(x) = H_n(x)\frac{e^{-x^2/2}}{\sqrt{N_n}} \tag{44}$$

24. [**1**] Show that the $\varphi_n(x)$ are normalized eigenfunctions for $\hat{H}$ and give their eigenvalues E_n.

The thermal density matrix $\rho(x,y,\beta)$ describes the quantum statistical features of $\hat{H}$. It satisfies to the following diffusion equation

$$\frac{\partial \rho}{\partial \beta} = -\hat{H}_x \rho \tag{45}$$

in which $\hat{H}_x$ means that $\hat{H}_x$ acts on the x-variable only. It is then natural to look for a solution that is an expansion over the φ_n basis:

$$\rho(x,y,\beta) = \sum_{n=0}^{\infty} c_n(\beta)\varphi_n(x)\varphi_n(y) \ . \tag{46}$$

25. [**1**] Show that $c_n(\beta) = e^{-\beta/2}e^{-\beta n}$.

For $0 \le t < 1$, Mehler's formula reads

$$\sum_{n=0}^{\infty} t^n \varphi_n(x)\varphi_n(y) = \frac{1}{\sqrt{\pi(1-t^2)}} \exp\left(\frac{t}{1-t^2}2xy - \frac{1+t^2}{1-t^2}\frac{x^2+y^2}{2}\right) \tag{47}$$

$$= \frac{1}{\sqrt{\pi(1-t^2)}} \exp\left(\frac{x^2-y^2}{2} - \frac{(x-yt)^2}{1-t^2}\right) \tag{48}$$

26. [**1**] After explaining the Gaussian equality $e^{-x^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2+2ixu}\mathrm{d}u$, show that

$$H_n(x) = \frac{(-2i)^n}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{\infty} u^n e^{-u^2+2ixu}\mathrm{d}u \tag{49}$$

27. [**2**] Using (49), prove (48) through an explicit resummation of the left-hand side expansion.

28. [**2**] Finally, show that the thermal density matrix can be put into the compact form

$$\rho(x,y,\beta) = C(\beta)\exp\left(-A_+(\beta)\frac{(x+y)^2}{4} - A_-(\beta)\frac{(x-y)^2}{4}\right) \tag{50}$$

    in which C and the $A_\pm$ are simple functions of β to be determined explicitly.