# Exam on Mathematical tools 

Wednesday January 12th

You are allowed to use only your notes and the documents distributed during the lectures. If you think there is an error in the subject, explain it on your copy and proceed with the assignment.
As a guide, we give an estimate of points [ $\boldsymbol{n}$ ] for each question (the final mean of the class will be rescaled so these are indicative only).

## Poisson formula on the disk [7]

We consider the unit circle $\mathcal{C}$ with the set of points $|z|=1$ for complex $z$.

## 1. From Cauchy to Poisson

a) [1] Let $z_{0}$ be a point within the unit disk $\mathcal{D}$ of boundary $\mathcal{C}$ and $f(z)$ some analytic function over the complex plane. Apply Cauchy integral formula to express $f\left(z_{0}\right)$ in terms of an integral over $\mathcal{C}$.
b) [2] For a point located at $1 / \bar{z}_{0}$ (inverse of the conjugate of $z_{0}$ ), compute $\oint_{\mathcal{C}} \frac{f(z)}{z-1 / \bar{z}_{0}} \mathrm{~d} z$.
c) [2] Infer the Poisson circle formula

$$
\begin{equation*}
f\left(z=r e^{i \theta}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{1-r^{2}}{1-2 r \cos (\theta-\phi)+r^{2}} f\left(e^{i \phi}\right) \mathrm{d} \phi \tag{1}
\end{equation*}
$$

2. [2] Application to electrostatics: we consider the electrostatic potential $V(r, \theta)$ in polar coordinates over the disk $\mathcal{D}$ and satisfying to the Poisson equation $\hat{\Delta} V=0$ in the absence of charges. The boundary conditions are that $V(1, \theta)=v(\theta)$ a given function. Give an expression of the solution $V(r, \theta)$ as a function of $v(\theta)$ by giving supporting arguments to your formula.

## Saddle point technique with many variables [8]

We consider the function $f(x)$ of the variable $x>0$ defined under the integral form

$$
\begin{equation*}
f(x)=\int \mathrm{d} \vec{t} \exp (-x h(\vec{t})) \tag{2}
\end{equation*}
$$

assuming a entire multivariate real function $h(\vec{t})$ of $d$ variables that has a single absolute minimum $\overrightarrow{t_{c}}$. We assume for sake of simplicity that $h$ is entire and that the integral range if the whole space $\mathbb{R}^{d}$ and we take the limit $x \rightarrow \infty$.

1. [3] Find the leading term of the asymptotic behavior of $f(x)$ using the notation $\mathbf{H}_{i j}^{c}=\left.\frac{\partial^{2} h}{\partial t_{i} \partial t_{j}}\right|_{\vec{t}_{c}}$ for the Hessian matrix.
2. [2] We now introduce

$$
\begin{equation*}
Z(x, \vec{b})=\int \mathrm{d} \vec{t} \exp [-x(h(\vec{t})-\vec{b} \cdot \vec{t})] \tag{3}
\end{equation*}
$$

Write the saddle point equation and the corresponding asymptotic behavior for $Z(x, \vec{b})$ as $x \rightarrow \infty$.
3. [3] Compute $\langle\vec{t}\rangle$ averaged over the distribution $p(x, \vec{t})=\exp [-x h(\vec{t})] / Z(x, \overrightarrow{0})$ to leading order in $x$ as a function of the saddle point parameters.

## Green's function for 1D and 2D waves [12]

We consider the d'Alembert operator Green's function in $d$-dimension obeying

$$
\begin{equation*}
\left[\hat{\Delta}_{\vec{r}}-\frac{\partial^{2}}{\partial t^{2}}\right] G\left(\vec{r}, \vec{r}^{\prime}, t, t^{\prime}\right)=\delta\left(\vec{r}-\vec{r}^{\prime}\right) \delta\left(t-t^{\prime}\right) \tag{4}
\end{equation*}
$$

with vanishing boundary conditions at infinity.

1. [1] What is the meaning of $t^{\prime}$ and $\vec{r}^{\prime}$ ? Argue that $G$ should be a function of $\vec{R}=\vec{r}-\vec{r}^{\prime}$ and $\tau=t-t^{\prime}$ only.

We take the following notations for the Fourier transforms

$$
\begin{equation*}
\tilde{G}(\vec{k}, \omega)=\int \mathrm{d} \vec{R} \int \mathrm{~d} \tau e^{-i(\vec{k} \cdot \vec{R}-\omega \tau)} G(\vec{R}, \tau), \quad G(\vec{R}, \tau)=\int \frac{\mathrm{d} \vec{k}}{(2 \pi)^{d}} \int \frac{\mathrm{~d} \omega}{2 \pi} e^{i(\vec{k} \cdot \vec{R}-\omega \tau)} \tilde{G}(\vec{k}, \omega) \tag{5}
\end{equation*}
$$

and we follow a slightly different approach then in the lectures. We'll note $R=\|\vec{R}\|$.
2. [2] Give the expression of $G(\vec{R}, \tau)$ in terms of a Fourier integral and using $k=\|\vec{k}\|$.
3. [2] After properly regularizing the integral over $\omega$, show that one can write

$$
\begin{equation*}
G(\vec{R}, \tau)=-\Theta(\tau) \int \frac{\mathrm{d} \vec{k}}{(2 \pi)^{d}} \frac{\sin (k \tau)}{k} e^{i \vec{k} \cdot \vec{R}} \tag{6}
\end{equation*}
$$

with $\Theta$ the Heavyside function.
4. $d=1$ case
a) [1] Express the integral $\int_{-\infty}^{\infty} \frac{\sin (k y)}{k} \mathrm{~d} k$ using the $\Theta$ function.
b) [2] Finally recover the Green's function $G(x, \tau)$ in terms of a single $\Theta$ function.
5. $d=2$ case
a) [1] Rewrite Eq. (6) using polar coordinates ( $k, \theta$ ) in $k$-space.
b) [3] Using the Bessel function identity for real $\alpha$

$$
\begin{equation*}
J_{0}(\alpha)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathrm{~d} \theta e^{i \alpha \cos \theta}=\frac{2}{\pi} \int_{1}^{\infty} \mathrm{d} u \frac{\sin (\alpha u)}{\sqrt{u^{2}-1}} \tag{7}
\end{equation*}
$$

find the Green's function $G(R, \tau)$ in terms of a single $\Theta$ function and some explicit dependence in $R$ and $\tau$.

## Laguerre's polynomials [19]

We recall the table of the lecture notes but we did not prove all formulas for Laguerre polynomials $L_{n}(x)$ that are orthogonal polynomials in $x$ of order $n$.

| Differential equation | $x y^{\prime \prime}(x)+(1-x) y^{\prime}(x)+n y(x)=0$ |
| :--- | :---: |
| Rodrigues formula: $w(x)=e^{-x}$ | $L_{n}(x)=\frac{e^{x}}{n!} \frac{\mathrm{d}^{n}}{\mathrm{~d} x^{n}}\left(x^{n} e^{-x}\right)$ |
| Parameters | $S=\left[0, \infty\left[, \quad \lambda_{n}=n, \quad c_{n}=\frac{1}{n!}, \quad N_{n}=1\right.\right.$ |
| Generating function | $G(x, t)=\frac{e^{-x t /(1-t)}}{1-t}=\sum_{n=0}^{\infty} L_{n}(x) t^{n}$ |
| Recurrence relation | $(n+1) L_{n+1}(x)=(2 n+1-x) L_{n}(x)-n L_{n-1}(x)$ |

## General properties

1. [1] Explain through a calculation why the weight function $w(x)$ is an exponential.
2. [3] Generating function. Consider that you only know the definition $G(x, t)=\sum_{n=0}^{\infty} L_{n}(x) t^{n}$ and Rodrigues formula. Show that $G(x, t)=\frac{e^{-x t /(1-t)}}{1-t}$.
3. [2] Prove the recurrence relation using previous results

$$
\begin{equation*}
(n+1) L_{n+1}(x)=(2 n+1-x) L_{n}(x)-n L_{n-1}(x) \tag{8}
\end{equation*}
$$

4. [2] Show that

$$
\begin{equation*}
x L_{n}^{\prime}(x)=n\left(L_{n}(x)-L_{n-1}(x)\right) \tag{9}
\end{equation*}
$$

5. [1] Give the values for $L_{n}(0)$.
6. [1] What is the coefficient $a_{n}$ of the leading term $L_{n}(x)=a_{n} x^{n}+\ldots$ ?
7. [2] Give the explicit expression of $n!L_{n}(x)$ as a function of $x$ for $n=0,1,2,3$.
8. [1] Write down the explicit orthogonality relation of the $L_{n}$ polynomials using the $N_{n}$ coefficient.
9. [2] Show that $N_{n}=1$ with the help of Rodrigues formula.
10. Let $f(x)$ be a function that we expand over the $L_{n}(x)$ basis as $f(x)=\sum_{n=0}^{\infty} a_{n} L_{n}(x)$.
a) [1] Give the explicit formula allowing one to compute the $a_{n}$.
b) [2] Use this formula to compute the $a_{n}$ in the case where $f(x)=e^{-r x}$ with $r>0$.
c) [1] Check your result using the generating function.

## The end.

