Poisson formula on the disk [7]

1. a) [1] $f(z_0) = \dfrac{1}{2\pi i}\oint_{\mathcal{C}} \dfrac{f(z)}{z-z_0}\mathrm{d}z$.

   b) [2] Clearly, the point $1/\bar{z}_0$ lies outside the circle $\mathcal{C}$ so there is no pole inside the circle. We get $0 = \dfrac{1}{2\pi i}\oint_{\mathcal{C}} \dfrac{f(z)}{z-1/\bar{z}_0}\mathrm{d}z$.

   c) [2] We subtract the two previous relation in order to rewrite the integrals with real numbers and an angular variable only. We take the parametrization $z_0 = re^{i\theta}$ and $z = e^{i\phi}$, $\mathrm{d}z = iz\mathrm{d}\phi$: first, after calculation

$$\left(\frac{1}{z-z_0} - \frac{1}{z-1/\bar{z}_0}\right) iz = i\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}$$

   that is inserted into the integral to give the desired result

$$f(z=re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, f(e^{i\phi})\,\mathrm{d}\phi\,. \tag{1}$$

2. [2] Without charges, the electrostatic potential is an harmonic function. So are the real an imaginary parts of the previous f function since it is analytical. With $f(r,\theta) = f_R(r,\theta) + if_I(r,\theta)$, we thus have two properties : $\hat{\Delta} f_R = 0$ and $\hat{\Delta} f_I = 0$ and both $f_{R,I}$ satisfy to (1) by taking the real and imaginary part of the Poisson relation. Last, choosing say the real part such that $f_R(1,\theta) = v(\theta)$, we have that $V = f_R$ is solution of the Poisson equation and the value of the potential inside the disk is obtained from

$$V(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, v(\phi)\,\mathrm{d}\phi\,. \tag{2}$$

   involving only real numbers.

Saddle point technique with many variables [8]

We consider the function $f(x)$ of the variable $x > 0$ defined under the integral form

$$f(x) = \int \mathrm{d}\vec{t}\,\exp\left(-x\,h(\vec{t})\right) \tag{3}$$

assuming a entire multivariate real function $h(\vec{t})$ of d variables that has a single absolute minimum $\vec{t}_c$. We assume for sake of simplicity that h is entire and that the integral range if the whole space $\mathbb{R}^d$ and we take the limit $x \to \infty$.

1. [3] We expand h around its minimum $\vec{t}_c$ to second order

$$h(\vec{t}) \simeq h(\vec{t}_c) + \frac{1}{2}\sum_{i,j=1}^{d} \mathbf{H}^c_{ij}(t_i - t_{c,i})(t_j - t_{c,j})$$

   using the notations $\mathbf{H}^c_{ij} = \left.\dfrac{\partial^2 h}{\partial t_i \partial t_j}\right|_{\vec{t}_c}$ for the Hessian matrix that is a positively defined matrix for an absolute minimum (we assume no zero eigenvalue for non-degenerate minimum). The saddle point equation giving the extrema is

$$\vec{\nabla}_{\vec{t}}\, h|_{\vec{t}_c} = \vec{0} \tag{4}$$

   Then, the saddle point technique gives

$$f(x) \simeq \exp\left(-x\,h(\vec{t}_c)\right)\int \mathrm{d}\vec{t}\,\exp\left(-\frac{x}{2}\sum_{i,j=1}^{d}\mathbf{H}^c_{ij}(t_i - t_{c,i})(t_j - t_{c,j})\right) \tag{5}$$

   which is a Gaussian integral. We get

$$f(x) \simeq \sqrt{\frac{(2\pi)^d}{\det \mathbf{H}^c}}\,\frac{e^{-x\,h(\vec{t}_c)}}{x^{d/2}} \tag{6}$$

2. [2] The saddle point equation is now

$$\vec{\nabla}_{\vec{t}}\, h|_{\vec{t}_c} = \vec{b} \tag{7}$$

   so $\vec{t}_c(\vec{b})$. Applying the previous result using the fact that the external field $\vec{b}$ does not change the Hessian matrix, we get

$$Z(x,\vec{b}) \simeq \sqrt{\frac{(2\pi)^d}{\det \mathbf{H}^c(\vec{b})}}\,\frac{e^{-x\left[h(\vec{t}_c(\vec{b})) - \vec{b}\cdot\vec{t}_c(\vec{b})\right]}}{x^{d/2}} \tag{8}$$

3. **[3]** As usual,

$$\langle \vec{t}\,\rangle = \frac{1}{Z(x,\vec{0}\,)} \int \mathrm{d}\vec{t}\,\vec{t}\,\exp\left[-x\; h(\vec{t}\,)\right] \tag{9}$$

$$= \frac{1}{x\,Z(x,\vec{0}\,)} \vec{\nabla}_{\vec{b}} \int \mathrm{d}\vec{t}\,\exp\left(-x\;\left[h(\vec{t}\,) - \vec{b}\cdot\vec{t}\,\right]\right)\Big|_{\vec{b}=\vec{0}} \tag{10}$$

$$= \frac{1}{x} \vec{\nabla}_{\vec{b}} \ln Z(x,\vec{b}\,)\Big|_{\vec{b}=\vec{0}} \tag{11}$$

Using the saddle point approximation, we get

$$\frac{1}{x}\ln Z(x,\vec{b}\,) \simeq -h(\vec{t}_c(\vec{b}\,)) + \vec{b}\cdot\vec{t}_c(\vec{b}\,) - \frac{d}{2x}\ln(x/2\pi) - \frac{1}{2x}\ln\det \mathbf{H}^c(\vec{b}\,)$$

The last two terms can be neglected when $x \to \infty$. The first two give

$$\langle \vec{t}\,\rangle \simeq -\vec{\nabla}_{\vec{b}} h(\vec{t}_c(\vec{b}\,)) + \vec{t}_c(\vec{b}\,) + \vec{b}\cdot\vec{\nabla}_{\vec{b}}\vec{t}_c(\vec{b}\,)$$

but by chain rule, we have $\vec{\nabla}_{\vec{b}} h = \vec{\nabla}_{\vec{t}} h \cdot \vec{\nabla}_{\vec{b}}\vec{t}_c(\vec{b}\,) = \vec{b}\cdot\vec{\nabla}_{\vec{b}}\vec{t}_c(\vec{b}\,)$ where we have used the saddle point equation in the last equality so that finally:

$$\langle \vec{t}\,\rangle \simeq \vec{t}_c(\vec{b}\,)$$

Green's function for 1D and 2D waves [12]

1. **[1]** t' and $\vec{r}\,'$ are the time and position of the impulse. In free space with translational and rotational invariance, we expect the Green's function to depend only on the relative distance $\vec{R} = \vec{r} - \vec{r}\,'$ (actually its norm) and the time from impulse $\tau = t - t'$.

We'll note $R = \|\vec{R}\|$.

2. **[2]** Fourier transforming the equation after changing variables to

$$\left[\hat{\Delta}_{\vec{R}} - \frac{\partial^2}{\partial\tau^2}\right] G(\vec{R},\tau) = \delta(\vec{R}\,)\delta(\tau) \qquad \text{gives} \qquad (\omega^2 - \vec{k}^{\,2})\tilde{G}(\vec{k},\omega) = 1 \tag{12}$$

so, with $k = \|\vec{k}\|$,

$$G(\vec{R},\tau) = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^d} \int \frac{\mathrm{d}\omega}{2\pi}\, \frac{e^{i(\vec{k}\cdot\vec{R}-\omega\tau)}}{\omega^2 - k^2} \tag{13}$$

3. **[2]** For the integral over ω, we have two poles $\omega = \pm k$ lying on the real axis. We regularize this integral using the causal regularization by shifting these poles below the real axis to $\omega = \pm k - i\varepsilon$, $\varepsilon \to 0^+$. Then, for $\tau < 0$, we compute the integral using the upper half-circle and residue theorem, which gives zero since there is no poles. For $\tau > 0$, we must use the lower half-circle. Adding the contribuion of the two poles, we get

$$G(\vec{R},\tau) = -\Theta(\tau)\int \frac{\mathrm{d}\vec{k}}{(2\pi)^d}\, \frac{\sin(k\tau)}{k}\, e^{i\vec{k}\cdot\vec{R}} \tag{14}$$

with Θ the Heavyside function, that accounts for vanishing of the integral when $\tau < 0$.

4. $d = 1$ **case**

a) **[1]** Using the result of the lecture / tutorial, we know that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(u)}{u}\mathrm{d}u = \pi$. Now, we the y variable, if we write $I(y) = \displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(ky)}{k}\mathrm{d}k$, we have $I(y) = 0$ if $y = 0$, using $u = ky$, $I(y) = +1$ if $y > 0$ and $I(y) = -1$ if $y < 0$ (it is the sign function !). In the end, we can use

$$I(y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(ky)}{k}\mathrm{d}k = 2\,\Theta(y) - 1$$

b) **[2]** First

$$G(x,\tau) = -\Theta(\tau)\int \frac{\mathrm{d}k_x}{2\pi}\,\frac{\sin(k\tau)}{k}\,\cos(k_x x) = -\Theta(\tau)\int\frac{\mathrm{d}k_x}{2\pi}\,\frac{\sin(k_x\tau)}{k_x}\,\cos(k_x x) \tag{15}$$

Using $\cos(k_x x)\sin(k_x\tau) = \frac{1}{2}[\sin(k_x(\tau + x)) + \sin(k_x(\tau - x))]$, we get

$$G(x,\tau) = -\frac{\Theta(\tau)}{4}[I(x+\tau) + I(\tau - x)] = -\frac{\Theta(\tau)}{2}[\Theta(x+\tau) + \Theta(\tau - x) - 1] \tag{16}$$

With $\tau > 0$, the last term can be shown to be equal to $\Theta(\tau - |x|)$ and then, the condition $\Theta(\tau)$ can get absorbed since non-zero result requires $\tau > |x| > 0$.

$$G(x,\tau) = -\frac{1}{2}\Theta(\tau - |x|) \tag{17}$$

5. $d = 2$ **case**
   a) **[1]** Eq. (14) in polar coordinates (k, θ) in k-space reads

$$G(R,\tau) = -\Theta(\tau)\int_0^\infty \int_0^{2\pi} \frac{k\mathrm{d}k\mathrm{d}\theta}{(2\pi)^2}\frac{\sin(k\tau)}{k}\,e^{ikR\cos\theta} = -\Theta(\tau)\int_0^\infty \frac{\mathrm{d}k}{2\pi}\,\sin(k\tau)\,J_0(kR) \tag{18}$$

   with the Bessel function $J_0(\alpha) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \mathrm{d}\theta\, e^{i\alpha\cos\theta}$

   b) **[3]** Using the Bessel function identity for real α

$$J_0(\alpha) = \frac{2}{\pi}\int_1^\infty \mathrm{d}u \frac{\sin(\alpha\, u)}{\sqrt{u^2-1}} \tag{19}$$

   we get

$$G(R,\tau) = -\frac{\Theta(\tau)}{2\pi}\int_1^\infty \frac{\mathrm{d}u}{\sqrt{u^2-1}}\underbrace{\frac{2}{\pi}\int_0^\infty \mathrm{d}k\,\sin(k\tau)\,\sin(kR\,u)}_{\delta(Ru-\tau)} = -\frac{\Theta(\tau)}{2\pi R}\int_{R-\tau}^\infty \mathrm{d}\xi \frac{\delta(\xi)}{\sqrt{((\xi+\tau)/R)^2-1}} \tag{20}$$

   if $R - \tau < 0$, 0 is not in the range of the integral so the result is zero. It gives an overall $Theta(\tau - R)$ factor that can be merged with the $\Theta(\tau)$. One eventually gets

$$G(R,\tau) = -\frac{\Theta(\tau - R)}{2\pi\sqrt{\tau^2 - R^2}} \tag{21}$$

   which somehow interpolates between the light-cone result in 1D and the pulse result in 3D.

Laguerre's polynomials [19]

General properties

1. **[1]** Applying the result (6.5) of the lecture, using $p(x) = x$ and $q(x) = 1 - x$ we get:

$$w(x) = \frac{1}{p(x)}\exp\left\{\int^x \frac{q(x')}{p(x')}\mathrm{d}x'\right\} = \frac{1}{x}\exp\left\{-\int^x \frac{1-x'}{x'}\mathrm{d}x'\right\} = \frac{1}{x}e^{\ln x - x} = e^{-x} \tag{22}$$

2. **[3] Generating function.** We use first **Schlaefli representation** and then resume the series under the integral. We start from the Cauchy integral formula with $f(z) = z^n e^{-z}$ that is holomorphic and taking $z_0 = x$:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^n e^{-x} = \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{z^n e^{-z}}{(z-x)^{n+1}}dz \tag{23}$$

   with $\mathcal{C}$ centered around x. Combining it with $L_n(x) = \dfrac{e^x}{n!}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}x^n e^{-x}$, one has

$$L_n(x) = \frac{e^x}{2i\pi}\oint_{\mathcal{C}} \frac{z^n e^{-z}}{(z-x)^{n+1}}\mathrm{d}z \tag{24}$$

   We inject this in the definition of the generating function, the countour is chosen such that $|zt/(z-x)| < 1$ to ensure convergence of the sum and to contain the $x - t$ pole:

$$G(x,t) = \frac{e^x}{2i\pi}\sum_{n=0}^\infty \oint_{\mathcal{C}} \frac{(zt)^n e^{-z}}{(z-x)^{n+1}}\mathrm{d}z = \frac{e^x}{2i\pi}\oint_{\mathcal{C}}\sum_{n=0}^\infty \left(\frac{zt}{z-x}\right)^n \frac{e^{-z}}{(z-x)}\mathrm{d}z \tag{25}$$

$$= \frac{e^x}{2i\pi}\oint_{\mathcal{C}} \frac{1}{1-\frac{zt}{z-x}}\frac{e^{-z}}{(z-x)}\mathrm{d}z = \frac{e^x}{2i\pi(1-t)}\oint_{\mathcal{C}} \frac{e^{-z}}{z - x/(1-t)}\mathrm{d}z \tag{26}$$

$$= \frac{e^x}{1-t}e^{-x/(1-t)} = \frac{e^{-xt/(1-t)}}{1-t} \tag{27}$$

3. **[2]** From $\dfrac{\partial}{\partial t}G(x,t) = \sum_{n=0}(n+1)L_{n+1}(x)t^n = \dfrac{1-t-x}{(1-t)^2}G(x,t)$, we get

$$(1-t)^2 \sum_{n=0}(n+1)L_{n+1}(x)t^n = (1-t-x)\sum_{n=0} L_n(x)t^n \tag{28}$$

and equalling t^n coefficients gives

$$(n+1)L_{n+1}(x) - 2nL_n(x) + (n-1)L_{n-1}(x) = (1-x)L_n(x) - L_{n-1}(x) \tag{29}$$

and finally

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \tag{30}$$

4. **[2]** From $\dfrac{\partial}{\partial x}G(x,t) = \sum_{n=0} L'_n(x)t^n = \dfrac{-t}{1-t}G(x,t)$, we immediately get

$$L'_n(x) - L'_{n-1}(x) = -L_{n-1}(x) \tag{31}$$

Then, differentiating (30) gives

$$(n+1)L'_{n+1} = -L_n + (2n+1-x)L'_n - nL'_{n-1} \tag{32}$$
$$(n+1)(L'_{n+1} - L'_n) = -L_n - xL'_n + n(L'_n - L'_{n-1}) \tag{33}$$
$$-(n+1)L_n = -L_n - xL'_n - nL_{n-1} \tag{34}$$

in which we have used (31); so that

$$xL'_n(x) = n(L_n(x) - L_{n-1}(x)) \tag{35}$$

5. **[1]** From the definition of the generating function, we have $L_n(x) = \dfrac{1}{n!}\dfrac{\partial^n G(x,t)}{\partial t^n}\Big|_{t=0}$.
For $x=0$, it simplifies to $L_n(0) = \dfrac{1}{n!}\dfrac{\partial^n}{\partial t^n}\dfrac{1}{1-t}\Big|_{t=0} = \dfrac{1}{n!}\dfrac{n!}{(1-t)^{n+1}}\Big|_{t=0} = 1$.

6. **[1]** We have $a_0 = 1$ and from (30), it is clear that

$$(n+1)a_{n+1} = -a_n \quad \Longrightarrow \quad a_n = \frac{(-1)^n}{n!} \tag{36}$$

7. **[2]** We obtain from (30) that

$$L_0(x) = 1\ , \quad 1!\,L_1(x) = -x+1\ , \quad 2!\,L_2(x) = x^2 - 4x + 2\ , \quad 3!\,L_3(x) = -x^3 + 9x^2 - 18x + 6\ , \tag{37}$$

8. **[1]** Using the $w(x)$ weighted scalar product definition and notation N_n for the norm, it follows from the lectures that

$$\int_0^\infty e^{-x}L_n(x)L_m(x)\mathrm{d}x = N_n\delta_{nm} \tag{38}$$

9. **[2]** We have

$$N_n = \int_0^\infty e^{-x}L_n(x)L_n(x)\mathrm{d}x = \frac{1}{n!}\int_0^\infty L_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x})\mathrm{d}x = \frac{(-1)^n}{n!}\int_0^\infty \frac{\mathrm{d}^n L_n(x)}{\mathrm{d}x^n}x^n e^{-x}\mathrm{d}x \tag{39}$$

where we have used n times the integration by parts. Since L_n is a polynomial of order n, $\frac{\mathrm{d}^n L_n}{\mathrm{d}x^n}$ is just the constant $n!a_n = (-1)^n$ that we have computed. Finally,

$$N_n = \frac{1}{n!}\int_0^\infty x^n e^{-x}\mathrm{d}x = \frac{n!}{n!} = 1 \tag{40}$$

from the properties and definition of the Gamma function.

10. a) **[1]** We write $a_n = \langle L_n, f\rangle_w = \displaystyle\int_0^\infty e^{-x}L_n(x)f(x)\mathrm{d}x$

    b) **[2]** We proceed as for the computation of N_n.

$$a_n = \int_0^\infty e^{-(1+r)x}L_n(x)\mathrm{d}x = \frac{1}{n!}\int_0^\infty e^{-rx}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x})\mathrm{d}x = \frac{(-1)^n}{n!}\int_0^\infty \frac{\mathrm{d}^n e^{-rx}}{\mathrm{d}x^n}x^n e^{-x}\mathrm{d}x = \frac{r^n}{n!}\int_0^\infty x^n e^{-(1+r)x}\mathrm{d}x = \frac{r^n}{(1+r)^{n+1}}$$

    We eventually obtain the identity $e^{-rx} = \dfrac{1}{1+r}\displaystyle\sum_{n=0}^\infty \left(\frac{r}{1+r}\right)^n L_n(x)$.

    c) **[1]** simply use $r = t/(1-t)$ giving $t = r/(1+r)$ and use the definition of the generating function to recover the result.