Gaussian integrals

One dimension: $a > 0$, b real numbers

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2+bx} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}\,, \qquad \int_{0}^{\infty} dx\, x^n e^{-\frac{1}{2}ax^2} = \frac{1}{2}\left(\frac{2}{a}\right)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)$$

***d*-dimension:** $\vec{x} \in \mathbb{R}^d$, $\vec{b} \in \mathbb{R}^d$ two column vectors, $\mathbf{A}$ a $d \times d$ real matrix that is symmetric and positive (all eigenvalues are real and strictly positive):

$$Z(\mathbf{A}, \vec{b}\,) = \int_{\mathbb{R}^d} d\vec{x}\, \exp\left\{-\frac{1}{2}\vec{x}^{\mathsf{T}}\mathbf{A}\vec{x} + \vec{b}\cdot\vec{x}\right\} = \sqrt{\frac{(2\pi)^d}{\det \mathbf{A}}}\, \exp\left\{+\frac{1}{2}\vec{b}^{\mathsf{T}}\mathbf{A}^{-1}\vec{b}\right\} \tag{3.56}$$

$$\tilde{Z}(\mathbf{A}, \vec{b}\,) = \int_{\mathbb{R}^d} d\vec{x}\, \exp\left\{-\frac{1}{2}\vec{x}^{\mathsf{T}}\mathbf{A}\vec{x} + i\vec{b}\cdot\vec{x}\right\} = \sqrt{\frac{(2\pi)^d}{\det \mathbf{A}}}\, \exp\left\{-\frac{1}{2}\vec{b}^{\mathsf{T}}\mathbf{A}^{-1}\vec{b}\right\} \tag{3.57}$$

$$\bar{Z}(\mathbf{A}, \vec{b}\,) = \int_{\mathbb{R}^d} d\vec{x}\, \exp\left\{\frac{i}{2}\vec{x}^{\mathsf{T}}\mathbf{A}\vec{x} + i\vec{b}\cdot\vec{x}\right\} = \sqrt{\frac{(2\pi i)^d}{\det \mathbf{A}}}\, \exp\left\{-\frac{i}{2}\vec{b}^{\mathsf{T}}\mathbf{A}^{-1}\vec{b}\right\} \tag{3.58}$$

Wick's theorem: considering the Gaussian joint probability ($Z(\mathbf{A}) \equiv Z(\mathbf{A}, \vec{0}\,)$)

$$p(\vec{x}\,) = \frac{1}{Z(\mathbf{A})}\exp\left\{-\frac{1}{2}\vec{x}^{\mathsf{T}}\mathbf{A}\vec{x}\right\} \tag{3.59}$$

for an even number $2n$ of variables. If $\mathcal{P}(2n)$ is the set of partitions of $\{i_1, \ldots, i_{2n}\}$ in pairs:

$$\langle x_{i_1}\cdots x_{i_{2n}}\rangle = \sum_{P\,\in\,\mathcal{P}(2n)} \langle x_{i_{P(1)}} x_{i_{P(2)}}\rangle \cdots \langle x_{i_{P(2n-1)}} x_{i_{P(2n)}}\rangle \quad \text{with} \quad \langle x_i x_j\rangle = (\mathbf{A}^{-1})_{ij} \tag{3.60}$$

Asymptotic behavior of some integrals, saddle point methods

Asymptotic series: the asymptotic behavior of some function $f(z)$ when $|z| \to \infty$:

$$f(z) \sim \underbrace{L(z)}_{\text{leading term}} \times \underbrace{\sum_{n=0}^{\infty} \frac{a_n}{z^n}}_{\text{asymptotic series}} \tag{3.61}$$

Most of the time, these series are *divergent*. Good approximations are obtained when the series is truncated. A first method to derive the expansion is integration by parts when z is in a bound of the integral defining $f(z)$.

Steepest descent: let $h(t;x)$ be a function that diverges when $x \to \infty$ and has a single absolute **minimum** $t_c(x)$ in $[a,b]$, then the leading contribution for $x \to \infty$ is given by

$$f(x) = \int_a^b dt\ e^{-h(t;x)} \sim \sqrt{\frac{2\pi}{h_c''(x)}} e^{-h_c(x)}\ , \quad f(x) = \int_a^b dt\ g(t) e^{-x\,h(t)} \sim \sqrt{\frac{2\pi}{h_c''}} g(t_c)\, \frac{e^{-x\,h_c}}{\sqrt{x}}$$

where $h_c(x) \equiv h(t_c(x);x)$ and $h_c''(x) \equiv h''(t_c(x);x)$. Discussion of the validity and corrections is done case by case (several minima, boundary terms,...).

Stationary phase approximation: with basically the same properties for $h(t,x)$, one has to look for all extrema (both minima and maxima), with a $-$ sign for a maximum and a $+$ sign for a minimum, we have for each contribution when $x \to \infty$

$$f(x) = \int_a^b dt\ e^{ih(t;x)} \sim \sqrt{\frac{2\pi}{|h_c''(x)|}} e^{i(h_c(x) \pm \frac{\pi}{4})}\ ,\ f(x) = \int_a^b dt\ g(t) e^{ix\,h(t)} \sim \sqrt{\frac{2\pi}{|h_c''|}} g(t_c) \frac{e^{i(x\,h_c \pm \frac{\pi}{4})}}{\sqrt{x}}$$

Laplace's method: assuming that $h(t)$ has a single absolute minimum at t_c and close to which ($t \to t_c^+$) we have the following expansions

$$h(t) = h_c + \sum_{n=0}^{\infty} a_n (t-t_c)^{\mu+n} \qquad a_0 \neq 0\,,\ \mu > 0 \tag{3.62}$$

$$g(t) = \sum_{n=0}^{\infty} b_n (t-t_c)^{\beta-1+n} \qquad b_0 \neq 0\,,\ \beta > 0 \tag{3.63}$$

Then (beware of the lower bound in the integral), for $x \to \infty$

$$f(x) = \int_{t_c}^b dt\ g(t) e^{-x\,h(t)} \sim e^{-x\,h_c} \sum_{n=0}^{\infty} \Gamma\left(\frac{\beta+n}{\mu}\right) \frac{c_n}{x^{(\beta+n)/\mu}}$$

in which the c_n can be expressed as a function of the a_n, b_n, μ and β. For example

$$c_0 = \frac{b_0}{\mu a_0^{\beta/\mu}}\ , \quad c_1 = \frac{1}{\mu a_0^{(\beta+1)/\mu}} \left[b_1 - \frac{\beta+1}{\mu} \frac{a_1 b_0}{a_0} \right] \tag{3.64}$$

$$c_2 = \frac{1}{\mu a_0^{(\beta+2)/\mu}} \left[b_2 - \frac{\beta+2}{\mu} \frac{a_1 b_1}{a_0} + \left((\beta+\mu+2) a_1^2 - 2\mu a_0 a_2 \right) \frac{\beta+2}{2\mu^2} \frac{b_0}{a_0^2} \right] \tag{3.65}$$