Exercises on Green's function

✍ ***Exercise 4.1 Reciprocity relation***

We consider the Green function of an Hermitian operator $\hat{A}_{\vec{r}}$. Show the reciprocity formula:

$$G(\vec{r},\vec{r}\,') = G^*(\vec{r}\,',\vec{r}) \tag{4.1}$$

Give the operator interpretation of this relation (or matrix equivalent).

✍ ***Exercise 4.2 A simple example with boundary conditions***

We consider the Helmoltz operator in the one dimensional line $[0,1]$ with homogeneous boundary conditions for the Green function $G(0,x') = G(1,x') = 0$. The corresponding differential equation is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + m^2\phi(x) = \rho(x) \qquad \text{with} \qquad \phi(0) = \phi(1) = 0 \tag{4.2}$$

1. We reconsider first the homogeneous equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + m^2\phi(x) = 0 \tag{4.3}$$

recall the general form of the solution (without specifying the boundary conditions). We now take into account the specified boundary conditions. What are the eigenvalues λ_n of the Helmoltz operator?

2. We now would like to compute the Green function $G(x,x')$. We consider x' fixed and split the domain into $[0,x'[$ and $]x',1]$. Show that the derivative of the Green function is discontinuous when $x \to x'$ (assuming that $G(x,x)$ remains finite):

$$\lim_{\varepsilon\to 0} G'(x'+\varepsilon,x') - G'(x'-\varepsilon,x') = 1 \tag{4.4}$$

while G is continuous at $x = x'$.

3. We consider that

$$G(x,x') = \begin{cases} g_<(x) & \text{for} \quad x \le x' \\ g_>(x) & \text{for} \quad x \ge x' \end{cases} \tag{4.5}$$

Show that (and a similar form for $g_>(x)$)

$$g_<(x) = \frac{\sin(mx)\sin(m(x'-1))}{m\sin m} \tag{4.6}$$

4. By considering the trace of the operator $\hat{G}$, infer the following relation

$$\cot m = \frac{1}{m} + \sum_{n=1}^{+\infty} \frac{2m}{m^2-(\pi n)^2}\ , \ \text{where} \ \cot z = \frac{\cos z}{\sin z} \tag{4.7}$$

✍ ***Exercise 4.3 Diffusion equation***

Find the Green function of the diffusion equation in infinite system with homogeneous boundary conditions $\phi(\vec{r}) = 0$ at $\vec{r} \to \infty$ in dimension $d = 3$:

$$\frac{\partial\phi}{\partial t} = D\nabla^2\phi \tag{4.8}$$