## Chapter 2

## Continuous group - Lie algebra

### 2.1 Groups

### 2.1.1 Some generalities

A group is an abstract concept, essentially a set equipped with a multiplication law. In physics, it is a key concept as it often provides an underlying structure to the observations and equations describing the system. In particular, the existence of a group structure associated to the symmetries of a system implies many constraints on observable quantities.

Definitions a set $G$ is a group equipped with the multiplication law $\circ$ provided that

- for $a, b \in G, a \circ b=c \in G$.
- associativity : $(a \circ b) \circ c=a \circ(b \circ c)$.
- there exists a neutral element $e$ called identity such that for all $a \in G, a \circ e=e \circ a=a$.
- for all $a \in G$, there exists an inverse $a^{-1} \in G$, such that $a \circ a^{-1}=a^{-1} \circ a=e$.


## Types of groups

- Finite groups : finite number of elements in $G$.

Ex: parity, reflections symmetries, ponctual groups, permutations, cyclic group, ...

- Discrete groups : countable number of elements, thought there could be an infinite number Ex: translations on an infinite lattice, $\ldots$
- Continuous (or Lie) groups : uncountable number of elements, usually parametrized by continuous parameters (real of complex). Ex: continuous rotations, continuous translations, ...

Commutativity a group is commutative (abelian) if for all $a, b \in G$ one has $a \circ b=b \circ a$. Otherwise it is non-commutative (non-abelian).

## Some properties

- the neutral element and inverse of all elements are unique
- $(a \circ b)^{-1}=b^{-1} \circ a^{-1}$

Equivalence relation $a=t^{-1} \circ b \circ t, a$ is the transform of $b$ through $t, a$ and $b$ are conjugate.

The cyclic group $\mathbb{Z}_{n}$ : all $n$ elements can be generated from one called the generator $g$ such that $g^{n}=e$, ie. $\mathbb{Z}_{n}=\left\{g^{k}\right\}_{k=0, n-1}$.

Representations of a group: They define the action of a group on a given vector space denoted by $V$. In physics, objects belonging to the vector space $V$ are typically vectors, tensors or functions, and we work with linear representations. In this case, the representations are matrices of size $d \times d$ where $d=\operatorname{dim} V$ or linear differential operators depending on the nature of the physical object.

There exist many representations of a given group, as many as object to act on. One must not confuse the dimension of $V$ and the dimension of the group. For $g_{1}, g_{2} \in G$, the representations $D\left(g_{1}\right)$ and $D\left(g_{2}\right)$ must reproduce the multiplication law of the group, ie.

- $D\left(g_{1} \circ g_{2}\right)=D\left(g_{1}\right) D\left(g_{2}\right)$
- $D(e)=\mathbb{I}$
- $D\left(g^{-1}\right)=D(g)^{-1}$

Therefore, the $D(g)$ usually belong to matrix groups.

Lie groups: These are continuous groups such that: let $\vec{\alpha}$ be a $n$-dimensional parameter of $\Omega \subset \mathbb{R}^{n}$ and $g(\vec{\alpha}) \in G$, then $g(\vec{\alpha}) g(\vec{\beta})=g(\vec{\gamma}) \in G$ with $\vec{\gamma}=\phi(\vec{\alpha}, \vec{\beta})$ an analytic function. Similarly, $g^{-1}(\vec{\alpha})=$ $g\left(\vec{\alpha}^{\prime}\right) \in G$ with $\vec{\alpha}^{\prime}=f(\vec{\alpha})$ an analytic function. In general, one takes $g(\overrightarrow{0})=e$.

### 2.1.2 Some famous groups of matrices

The multiplication operator is simply the standard matrix multiplication.
$\mathbf{G L}(n, \mathbb{R})$ and $\mathbf{G L}(n, \mathbb{C})$ General linear group: group of real or complex invertible matrices of dimension $n \times n$.
$\mathbf{S L}(n, \mathbb{R})$ and $\mathbf{S L}(n, \mathbb{C})$ Special linear groups: subgroups of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ with matrices with determinant 1 , corresponding to volume and direction preserving transformations.
$\mathbf{O}(n)$ Orthogonal group: group of real orthogonal matrices of dimension $n \times n$ satisfying to $O O^{\top}=$ $O^{\top} O=1$, corresponding to distance preserving transformations that preserve a fixed point.
$\mathbf{S O}(n)$ Special Orthogonal group: group of orthogonal matrices of determinant one $\operatorname{det}(O)=1$, corresponding to the subgroup of $\mathrm{O}(n)$, also called rotation group.
$\mathbf{U}(n)$ Unitary group: group of complex unitary matrices of dimension $n \times n$ satisfying to $U U^{\dagger}=$ $U^{\dagger} U=1$.
$\mathbf{S U}(n)$ Special Unitary group: subgroup of $\mathrm{U}(n)$ corresponding to the group of unitary matrices of determinant one $\operatorname{det}(U)=1$.

### 2.2 Lie algebra

For continuous groups, one can study infinitesimal transformations, which introduces the notion of infinitesimal generators. The Lie algebra is the vector space spanned by these generators equipped with the Lie bracket defined below, which is the commutator in physics. We write $n$ the numbers of generators $(a=1, \ldots, n)$ which is then the dimension of the Lie algebra and differs from the dimension of the group.

Generators of a Lie group: by differentiation of the action of the group, one can write for small $\vec{\alpha}$

$$
\begin{equation*}
D(\vec{\alpha}) \simeq 1-i \vec{\alpha} \cdot \vec{T} \quad \text { or } \quad T_{a}=-\left.i \frac{\partial D}{\partial \alpha_{a}}\right|_{\vec{\alpha}=\overrightarrow{0}} \tag{2.1}
\end{equation*}
$$

in which $T_{a}$ are the $n$ generators. For a finite transformation $D(\vec{\alpha})=e^{-i \vec{\alpha} \cdot \vec{T}}$. In physics, $D$ is usually a unitary transform so that the $T_{a}$ are hermitian operators.

Lie bracket and structure constants The [,] symbol is called the Lie bracket and corresponds to the commutator for application in physics. The generators $T_{a}$ of the Lie algebra satisfy to the relations

$$
\begin{equation*}
\left[T_{a}, T_{b}\right]=i f_{a b c} T_{c} \tag{2.2}
\end{equation*}
$$

The coefficients $f_{a b c}$ are called the structure constants. The generators satisfy to the Jacobi identity

$$
\begin{equation*}
\left[T_{a},\left[T_{b}, T_{c}\right]\right]+\left[T_{b},\left[T_{c}, T_{a}\right]\right]+\left[T_{c},\left[T_{a}, T_{b}\right]\right]=0 \tag{2.3}
\end{equation*}
$$

### 2.3 Groups associated to rotations in 3D space

### 2.3.1 SO(3)

## Generators on 3D vectors:

$$
J_{1}=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{2.4}\\
0 & 0 & -i \\
0 & i & -1
\end{array}\right) \quad J_{2}=\left(\begin{array}{ccc}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{array}\right) \quad J_{3}=\left(\begin{array}{ccc}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Lie algebra so(3): $\left[J_{i}, J_{j}\right]=i \varepsilon_{i j k} J_{k}$
Generators on functions: $\vec{J}=-i \vec{x} \wedge \vec{\nabla}$
Rotation operator of angle $\psi$ around direction $\vec{n}: \quad R_{\vec{n}}(\psi)=e^{-i \psi \vec{n} \cdot \vec{J}}$

### 2.3.2 $\mathbf{S U}(2)$

## Generators on 2D complex vectors:

$J_{i}=\frac{\sigma_{i}}{2}$ with Pauli matrices $\sigma_{1}=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right), \sigma_{2}=\left(\begin{array}{cc}0 & -i \\ i & 0\end{array}\right), \sigma_{3}=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$
Lie algebra su(2): $\quad\left[J_{i}, J_{j}\right]=i \varepsilon_{i j k} J_{k}$
Rotation operator of angle $\psi$ around direction $\vec{n}$ :

$$
\begin{equation*}
U_{\vec{n}}(\psi)=e^{-i \psi \vec{n} \cdot \vec{J}}=\cos \left(\frac{\psi}{2}\right) \mathbb{I}-i \sin \left(\frac{\psi}{2}\right) \vec{n} \cdot \vec{\sigma} \tag{2.5}
\end{equation*}
$$

## Some properties of Pauli matrices:

$$
\begin{align*}
\sigma_{i}^{\dagger} & =\sigma_{i}, \quad \sigma_{i}^{2}=\mathbb{I}, \quad \operatorname{Tr}\left(\sigma_{i}\right)=0, \quad \operatorname{det}\left(\sigma_{i}\right)=-1  \tag{2.6}\\
\sigma_{i} \sigma_{j} & =\delta_{i j} \mathbb{I}+i \varepsilon_{i j k} \sigma_{k}  \tag{2.7}\\
(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) & =(\vec{a} \cdot \vec{b}) \mathbb{I}+i(\vec{a} \wedge \vec{b}) \cdot \vec{\sigma} \tag{2.8}
\end{align*}
$$

with $\vec{a}, \vec{b}$ two 3 -dimensional real vectors.

## Exercises on group theory

## Exercise 2.1 Properties of Pauli matrices

1. Show that $\left\{\sigma_{i}, \sigma_{j}\right\}=2 \delta_{i j} \mathbb{I}$.
2. Infer relation (2.7) and then (2.8).
3. Show relation (2.5).
4. By considering the scalar product $(M, N)=\frac{1}{2} \operatorname{Tr}\left(M^{\dagger} N\right)$ on $2 \times 2$ complex matrices, show that any $H$ complex matrix can be written, using the convention $\sigma_{0}=\mathbb{I}$ :

$$
\begin{equation*}
H=\sum_{i=0}^{3} \frac{1}{2} \operatorname{Tr}\left(\sigma_{i} H\right) \sigma_{i} \tag{2.9}
\end{equation*}
$$

5. We introduce the matrix $C=i \sigma_{2}=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$.
a) show that $C^{-1} \sigma_{i}^{\top} C=-\sigma_{i}$
b) for $U$ a $\mathrm{SU}(2)$ matrix, show that $U^{\dagger}=U^{-1}=C^{-1} U^{\top} C$
c) Infer that any matrix $U \in \mathrm{SU}(2)$ can be written

$$
\begin{equation*}
U=u_{0}-i \vec{u} \cdot \vec{\sigma} \tag{2.10}
\end{equation*}
$$

with $\left(u_{0}, \vec{u}\right) 4$ real parameters such that $u_{0}^{2}+\vec{u}^{2}=1$.
d) Find the connection between $\left(u_{0}, \vec{u}\right)$ and the general rotation parameters $(\vec{n}, \psi)$.

