Exercises on Linear algebra

✍ ***Exercise 3.1 Dyson equation from Jacobi iterative method***

We would like to have a perturbation expansion for the inverse of a matrix $\mathbf{A}$. We assume that $\mathbf{A}$ is invertible, with the notation $\mathbf{G} = \mathbf{A}^{-1}$, and that it can be decomposed as $\mathbf{A} = \mathbf{G}_0^{-1} - \mathbf{\Sigma}$, in which $\mathbf{G}_0$ is an invertible matrix that we known explicitly (we can choose it) and $\mathbf{\Sigma} = \mathbf{G}_0^{-1} - \mathbf{A}$ is the rest.

1. Show that $\mathbf{G}$ satisfies to the following self-consistent equation

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_0\mathbf{\Sigma}\mathbf{G}\ . \tag{3.1}$$

2. Propose an iterative scheme to get approximations of $\mathbf{G}$. Write down the first three terms of the perturbative expansion giving $\mathbf{G}$ as a function of $\mathbf{G}_0$ and $\mathbf{\Sigma}$.

✍ ***Exercise 3.2 Some formulas involving determinants***

For $\mathbf{A}$ an hermitian and positively defined matrix (eigenvalues $\lambda_i > 0$).

1. show that $\ln(\det(\mathbf{A})) = \mathrm{Tr}(\ln(\mathbf{A}))$, and $\det(\exp(\mathbf{A})) = \exp(\mathrm{Tr}(\mathbf{A}))$.
2. we consider that the matrix $\mathbf{A}$ is a function of the variable x, prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}\det(\mathbf{A}) = \det(\mathbf{A})\,\mathrm{Tr}\left(\mathbf{A}^{-1}\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}x}\right) \tag{3.2}$$

3. Infer the following relation, where a_{ij} are the coefficients of $\mathbf{A}$:

$$\frac{\partial}{\partial a_{ij}}\ln(\det(\mathbf{A})) = \left(\mathbf{A}^{-1}\right)_{ji} \tag{3.3}$$

✍ ***Exercise 3.3 Baker-Hausdorff and Trotter formulas***

1. Show that

$$e^{-\mathbf{T}}\mathbf{A}e^{\mathbf{T}} = \mathbf{A} + [\mathbf{A},\mathbf{T}] + \frac{1}{2!}[[\mathbf{A},\mathbf{T}],\mathbf{T}] + \frac{1}{3!}[[[\mathbf{A},\mathbf{T}],\mathbf{T}],\mathbf{T}] + \dots \tag{3.4}$$

2. and for $t \to 0$

$$e^{t(\mathbf{A}+\mathbf{B})} \simeq e^{t\mathbf{A}}e^{t\mathbf{B}} - \frac{t^2}{2}[\mathbf{A},\mathbf{B}] + \mathcal{O}(t^3) \tag{3.5}$$

✍ ***Exercise 3.4 Variational method for Hermitian matrices***

Let $\mathbf{A}$ be an $N \times N$ hermitian matrix of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$.
We define the Rayleigh quotient for $\vec{x} \neq 0$:

$$R(\vec{x}\,) = \frac{\vec{x}^{\,\dagger}\mathbf{A}\vec{x}}{\|\vec{x}\|^2} \tag{3.6}$$

1. Show that $\lambda_1 \leq R(\vec{x}\,) \leq \lambda_N$. Does it work for the matrix $\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$?
2. Give the expression of the gradient $\boldsymbol{\nabla}_{\vec{x}^\dagger}R(\vec{x}\,)$ as a function of $\mathbf{A}$, $\vec{x}$ and $R(\vec{x}\,)$. Comment.

✍ ***Exercise 3.5 Perron Fröbenius theorem in the case of symmetric positive matrices***
Let $\mathbf{A}$ be a real symmetric positive (i.e. $a_{ij} > 0$) matrix of size $N \times N$. We write λ_i the eigenvalues, ordered as $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ and $\vec{u}_i$ the corresponding normalized eigenvectors.
Prove the following statements (in this order since the first ones will induces the others):

(i) $\lambda_N > 0$.

(ii) $\vec{u}_N$ can be chosen such that its components are strictly positive $u_{i,N} > 0$.
Hints: use the variational principle to show first $u_{i,N} \geq 0$ and than consider the consequences of having one $u_{i,N} = 0$.

(iii) λ_N is non-degenerate.
Hints: use reductio ad absurdum

(iv) $\forall n \leq N-1$, we have $|\lambda_n| < \lambda_N$.
Hints: first show $|\lambda_n| \leq \lambda_N$ with the variational theorem and second, show that equality is absurd by considering $\lambda_1 = -\lambda_N$ and showing that the signs of components $u_{i,1}$ must vary.