Test on Complex analysis and Fourier transform

60mins

Thursday October 26th

You are allowed to use your notes and the summaries that I distributed.
The assignment is too long, do your best and don't mind about not doing everything.
The marking will take that into account.

The goal of this assignment is to prove and use the following formula, which often appears in physics:

$$S(a) = \sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a) \ , \qquad a \in \mathbb{R} \ , \quad n \in \mathbb{Z} \ \text{ and } \ \coth z = \frac{\cosh z}{\sinh z} \tag{1}$$

To do so, we are going to use two formulas that are more general and useful to compute series. In the last part, we give two nice applications of the formula. The three sections are totally independent and within each section, many questions are themselves independent (do not hesitate to jump if you are stuck somewhere).

Using the Poisson summation formula

We consider a function $g(x)$ of a real variable x, that is periodic of period 1. We recall that it can be expand in Fourier series, in the following way

$$g(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i2\pi n x} \ , \qquad \text{where } \ c_n = \int_{-1/2}^{1/2} g(x)\, e^{-i2\pi n x} dx \tag{2}$$

1. Give the expression of the Fourier transform $G(k)$ of $g(x)$, defined as $G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x)\, e^{-ikx} dx$, as a function of the c_n and delta functions.

2. We consider the Dirac comb $g(x) = \sum_{n=-\infty}^{+\infty} \delta(x-n)$. What is $G(k)$?

3. Let $f(x)$ be a function of a real variable x and $F(k)$ its Fourier transform. Infer the Poisson summation formula

$$\sum_{n=-\infty}^{+\infty} f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{+\infty} F(2\pi n) \tag{3}$$

4. We recall that we obtained that the Fourier transform of $f(x) = \frac{1}{x^2 + \sigma^2}$ is $F(k) = \sqrt{2\pi} \frac{e^{-\sigma |k|}}{2\sigma}$ ($\sigma > 0$ and within our convention for the Fourier transform). Use the Poisson summation formula to prove Eq. (1).

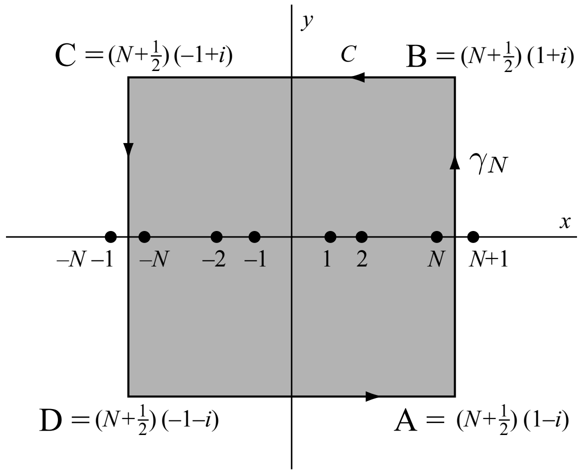

Figure 1: The contour γ_N. A, B, C and D are the corners of the contour and we give their coordinates.

Using the residue theorem

We consider the complex plane and the following square contour represented on Fig. 1 with $N \in \mathbb{N}^*$.

Bonus question (*skip it if too difficult and use the result*) We consider the complex function $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$. Show that this function is bounded on the straight lines AB and BC of γ_N, ie. $|\cot(\pi z)| \leq K$, where K is independent of N. By the symmetry $z \to -z$, it will be the same for CD and DA.

5. We consider a meromorphic function $f(z)$ defined over $\mathbb{C}$, with a finite number isolated poles $z_j \notin \mathbb{Z}$, and such that when $|z| \to \infty$, there exist a constant R such that $|f(z)| \leq \frac{R}{|z|^k}$ with $k > 1$. Show that

$$\lim_{N\to\infty} \oint_{\gamma_N} \pi \cot(\pi z)\, f(z)\, dz = 0\ . \tag{4}$$

6. Infer the following result (the notation $\mathrm{Res}(g(z), z_j)$ means the residue of $g(z)$ at z_j):

$$\sum_{n=-\infty}^{+\infty} f(n) = -\sum_j \mathrm{Res}(\pi \cot(\pi z)\, f(z), z_j) \tag{5}$$

7. Use this result to prove Eq. (1).

Applications

9. **calculation of Riemann sum**. Using (1), compute the following series: $\zeta(2) = \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$.

10. **Euler's identity**.
    a) We admit that Eq. (1) generalizes to $a \in \mathbb{C}$, show that

$$\cot t = \frac{1}{t} + \sum_{k=1}^{+\infty} \frac{2t}{t^2 - (\pi k)^2}\ , \qquad t \in \mathbb{R}\ , \qquad \text{and}\ \cot z = \frac{\cos z}{\sin z} \tag{6}$$

    b) By integrating the previous equality between 0 and x, $0 < x < \pi$, prove that

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right)\ , \tag{7}$$

which is used in path integral calculations, and which can be actually generalized to $\sin z$, $z \in \mathbb{C}$.