Test on Complex analysis and Fourier transform

60mins

Thursday October 25th

You are allowed to use your notes and the summaries that I distributed.
Many questions are independent and for the second exercice there is hardly no calculation to do.

Residues of order 2

1. Recall the formula allowing you to compute the residue of a pole z_0 of order p of some function $f(z)$.
2. Compute the following integral by choosing a good contour and specifying the poles and their order of the corresponding complex function:

$$I = \int_0^{+\infty} \frac{1}{(x^2+1)^2} dx \quad \text{and} \quad F(k) = \int_{-\infty}^{+\infty} \frac{e^{ikx}}{(x^2+a^2)^2} dx \qquad (a > 0) \tag{1}$$

Mittag-Leffler expansion

We know that a rational fraction can be decomposed as a sum of its simple elements, say $\frac{P(z)}{Q(z)} = \text{polynom} + \sum_k \sum_{n=1}^{p_k} \frac{c_{n,k}}{(z-z_k)^n}$ with $c_{n,k}$ some coefficients, z_k the poles associated to the zeroes of $Q(z)$ and p_k their order.

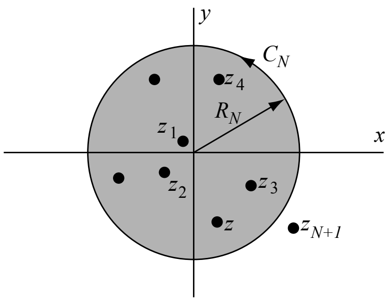

Figure 1: Sketch of the contour C_N of radius R_N.

We are going to see that a similar decomposition holds for meromorphic functions, that we denote by $f(z)$. For sake of simplicity, we use the following assumptions and notations:

(i) $f(z)$ has (up to an infinity of) *simple poles* (of order one) that we write z_k.

(ii) the residues associated to these poles are denoted by $r_k = \text{Res}(f(z), z_k)$.

(iii) these poles are labelled by ascending order and are all non-zero: $0 < |z_1| \leq |z_2| \leq \dots$

(iv) the first N poles can be contained in a circular contour C_N of radius R_N (see Fig. 1) that does not touch any pole.

(v) we assume that $\displaystyle\lim_{R_N \to \infty} \frac{1}{R_N} \max_{z \in C_N} |f(z)| = 0$.

1. We introduce the function $I(z) = \dfrac{1}{2i\pi} \oint_{C_N} \dfrac{f(\zeta)}{\zeta(\zeta - z)} d\zeta$ (z being inside C_N as in Fig. 1, $\forall k, z \neq z_k$). By applying the residue theorem, show that

$$I(z) = \sum_{k=1}^{N} \frac{r_k}{z_k(z_k - z)} + \frac{f(z)}{z} - \frac{f(0)}{z} \tag{2}$$

2. Taking the limit $R_N \to \infty$, prove that $f(z)$ can be expanded as

$$f(z) = f(0) + \sum_{k=1}^{\infty} r_k \left(\frac{1}{z - z_k} + \frac{1}{z_k} \right) \tag{3}$$

3. **Application:** by considering the function $f(z) = \frac{1}{\sin z} - \frac{1}{z}$ and after checking that it satisfies to the hypothesis, show the following two equalities

$$\frac{1}{\sin z} = \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{z - \pi k} = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - (\pi k)^2} \tag{4}$$

A connection with Weierstrass factorization theorem

We now consider an *entire* function $Z(z)$ that has only simple zeros z_k with $z_k \neq 0$. In particular, it means that, focusing on z_k, one can rewrite $Z(z) = (z - z_k) g_k(z)$ with $g_k(z_k) \neq 0$ and $g_k(z)$ an holomorphic function.

4. Show that the function $f(z) = Z'(z)/Z(z)$ has simple poles at z_k

5. Assuming that $f(z)$ obeys the hypothesis (i-v), show that one can factorize

$$Z(z) = Z(0) e^{Kz} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k} \right) \tag{5}$$

   with K a constant to be specified as a function of the z_k and $f(0)$. This constitutes a particular case of Weierstrass factorization theorem. *Consider integrating along a path in the complex plane from $z = 0$ to z through a line not passing through any pole.*

6. **Application:** By considering the function $Z(z) = \frac{\sin z}{z}$, show that

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2 \pi^2} \right) \tag{6}$$

Bonus: discussion on Fisher's zeros of partition functions

In 1964, Michael Fisher considered interpreting the non-analyticity of the free energy from the behavior the poles of its generalization to the complex plane $F(z) = -\frac{1}{\beta} \ln Z(z)$, in which the partition function of a system of energies E_n can be extended to the complex plane by defining

$$Z(z) = \sum_{n} e^{-zE_n}, \quad z \in \mathbb{C} \tag{7}$$

Along the real axis, one usually writes $z = \beta \in \mathbb{R}$ the inverse temperature. $Z(z)$ has zeros that are denoted z_k in the complex plane. We recall that phase transitions are signalled by non-analytic singularities of the free energy as a function of temperature.

7. Where are non-analytical points of $F(z)$ in the complex plane? *We assume* (5) *applies.*

8. Can $z_k \in \mathbb{R}$ for a finite and discrete spectrum associated with a finite system?

9. Do you see a way to reconcile the existence of phase transitions and the answers to the first two questions? One can draw a graph to help explain the emergence of phase transitions.