

Yet another Fresnel integral:

$$F(x) = \int_0^{2\pi} \ln(x^2 - 2x \cos \theta + 1) d\theta$$

1. a. $\ln(x^2 - 2x \cos \theta + 1) = \ln x^2 + \ln\left(\frac{1}{x^2} - 2 \cos \theta \frac{1}{x} + 1\right) \Rightarrow F(x) = 2\pi \ln x^2 + \int_0^{2\pi} \ln\left(\frac{1}{x^2} - 2 \cos \theta \frac{1}{x} + 1\right) d\theta$

so $F(x) = F(1/x) + 4\pi \ln x \quad (x \neq 0)$

b. $F(0) = \int_0^{\infty} \ln 1 = 0$; $F(1)$ satisfies to $F(1) = F(1) + 4\pi \ln 1$ but is undetermined so far.

a set $x = 1 + \varepsilon$, $\varepsilon \rightarrow 0^+$ gives $F(1+\varepsilon) = F(1-\varepsilon) + 4\pi \ln(1+\varepsilon) \rightarrow 0$
 $\frac{1}{x} \approx 1 - \varepsilon$
 so F is continuous at 1

for $x \rightarrow 0$, $\ln(1 - 2 \cos \theta x + x^2) \approx -2 \cos \theta x + O(x^2) \Rightarrow F(x) \approx 0 + O(x^2)$
 so F is continuous at 0.

2. a. if $x = 1$, the problem is that the minimum value $x^2 - 2x + 1 = (x-1)^2$ vanishes so that the behavior has to be controlled.

if $x \neq 1$, we can safely take the derivative under the integral,

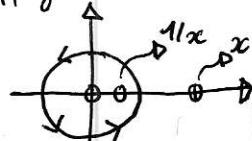
$$F'(x) = f(x) = \int_0^{2\pi} \frac{2(x - \cos \theta)}{x^2 - 2x \cos \theta + 1} d\theta \quad (\text{always } > 0)$$

as usual, we set $\begin{cases} z = e^{i\theta} \\ \theta \in [0, 2\pi] \end{cases}$; $dz = iz d\theta$; $x^2 - 2x \cos \theta + 1 = (x - e^{i\theta})(x - \bar{e}^{i\theta}) = (x - z)(x - \frac{1}{z})$

$$x - \cos \theta = x - \frac{1}{2}(e^{i\theta} + \bar{e}^{i\theta}) = x - \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2z}(2xz - z^2 - 1) = -(z-x)\left(z - \frac{1}{z}\right)\frac{x}{z}$$

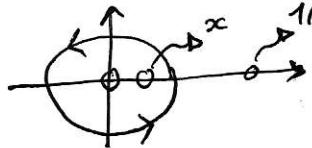
so
$$\boxed{f(x) = \frac{1}{iz} \oint_{|z|=1} \frac{z^2 - 2xz + 1}{z(z-x)(z-1/x)} dz}$$

b. we apply the residue theorem: for $x > 1$, there are two poles $z=0, 1/x$ inside the unit circle, we get



$$\boxed{f(x) = \frac{2i\pi}{iz} \left(\underbrace{\frac{+1}{-x(-1/x)}}_{=+1} + \underbrace{\frac{1/x^2 - 2 + 1}{1/x(1/x - x)}}_{=+1} \right) = \frac{4\pi}{x}}$$

c) we apply again the residue theorem: for $x < 1$ the two poles inside are $z = 0, \infty$ this time



$$\boxed{f(x) = \frac{2i\pi}{ix} \left(1 + \underbrace{\frac{x^2 - 2x + 1}{x(x-1/x)}}_{=-1} \right) = 0!}$$

the poles cancel each other.

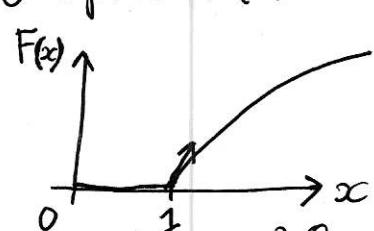
d) finally, using the continuity features of F we have

. $F(x) = \text{cste over }]0, 1[$ and $F(0) = 0 \Rightarrow F(x) = 0$ over $[0, 1]$
by continuity,

. $F(x) = 4\pi \ln x + \text{cste over }]1, \infty[$ and $F(1) \stackrel{?}{=} 0$ implies $F(x) = 4\pi \ln x$

finally

$$\boxed{F(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 1 \\ 4\pi \ln x & \text{for } x > 1 \end{cases}}$$



3. for $x=1$, we see that $x^2 - 2x \cos \theta + 1 = 2(1 - \cos \theta) = 4 \sin^2 \frac{\theta}{2}$

$$\text{or } F(1) = 0 = \int_0^{2\pi} \ln \left(4 \sin^2 \frac{\theta}{2} \right) d\theta = 2 \int_0^{2\pi} \ln \left(2 + \ln \left| \sin \frac{\theta}{2} \right| \right) d\theta = 4\pi \ln 2 + 2.2 \int_0^{\pi} \ln |\sin \theta| d\theta$$

$\varphi = \theta/2 \uparrow$

so we get

$$\boxed{\int_0^{\pi} \ln(\sin \theta) d\theta = -\pi \ln 2}$$

An academic example of the steepest descent method:

$$f(x) = \int_0^\infty \frac{t^x}{(1+t)^{x+1}} dt \quad \text{with } x > 1$$

1.a $h(t) = xt \ln(1+t) - x \ln t$ so that $f(x) = \int_0^\infty e^{-h(t)} dt$

$$h'(t) = \frac{dx}{1+t} - \frac{x}{t} = 0 \text{ for } t_c = 1/(x-1) ; 1+t_c = \frac{x}{x-1}$$

$$h''(t_c) = \frac{x}{t_c^2} - \frac{dx}{(1+t_c)^2} = \dots = x \frac{(x-1)^3}{x} > 0 \quad (\text{minimum})$$

$$h(t_c) = xc(\ln x - \ln(x-1) + \ln(x-1)) = xc(x \ln x - (x-1) \ln(x-1))$$

so we get, using the result of the lectures:

$$f(x) \sim \sqrt{\frac{2\pi x}{(x-1)^3}} e^{((x-1)\ln(x-1) - x \ln x)x}$$

b. we are in the standard situation where $h(t) = x \tilde{h}(t)$ (\tilde{h} independent of x)

so the width of the gaussian approximation scales

as $1/\sqrt{x}$ while the minimum is at fixed position so the boundary terms are not relevant.

2.a we need to map $t \in [0, \infty]$ to $u \in [0, 1]$ in an algebraic way, we take

$$u = \frac{t}{1+t} \Leftrightarrow t = \frac{u}{1-u} ; dt = \frac{du}{(1-u)^2} ; \frac{1}{1+t} = 1-u$$

so that

$$f(x) = \int_0^1 \frac{du}{(1-u)^2} \frac{u^x}{(1-u)^{x+1}} (1-u)^{x+1} = \int_0^1 du u^{x+1-1} (1-u)^{(x-1)x-1-1}$$

$$f(x) = B(x+1, (x-1)x-1) = \frac{\Gamma(x+1)\Gamma((x-1)x-1)}{\Gamma(x)}$$

b. we write Stirling formula as

$$\Gamma(x+1) \underset{x \rightarrow \infty}{\sim} \sqrt{2\pi} e^{x \ln x - x + \frac{1}{2} \ln x}$$

we get $\sqrt{2\pi}$ for the prefactor and in the exponential, we have with $f(x) = \sqrt{2\pi} e^{K(x)}$:

$$K(x) = x \ln x - x + \frac{1}{2} \ln x + ((\alpha-1)x-2) \ln((\alpha-1)x-2) - (\alpha-1)x + 2 + \frac{1}{2} \ln((\alpha-1)x-2) \\ - (\alpha x-1) \ln(\alpha x-1) + \alpha x - 1 - \frac{1}{2} \ln(\alpha x-1) \text{ of order } \frac{1}{x}$$

then, we have to be carefull with the logs:

$$\ln(\alpha x - 1) = \ln(\alpha x) + \ln(1 - \frac{1}{\alpha x}) \approx \ln(\alpha x) - \frac{1}{\alpha x}$$

$$\ln((\alpha-1)x-2) \approx \ln((\alpha-1)x) - \frac{2}{(\alpha-1)x}$$

$$\text{so } (\alpha x - 1) \ln(\alpha x - 1) \approx \alpha x \left(1 - \frac{1}{\alpha x}\right) \left(\ln(\alpha x) - \frac{1}{\alpha x}\right) \approx \alpha x \ln(\alpha x) - \ln(\alpha x) - 1$$

$$((\alpha-1)x-2) \ln((\alpha-1)x-2) \approx (\alpha-1)x \left(1 - \frac{2}{(\alpha-1)x}\right) \left(\ln((\alpha-1)x) - \frac{2}{(\alpha-1)x}\right)$$

$$\approx (\alpha-1)x \ln((\alpha-1)x) - 2 \ln((\alpha-1)x) - 2$$

so to order one, we can write

$$K(x) \approx x \ln x + (\alpha-1)x \left(\ln(\alpha-1) + \ln x \right) - \alpha x \left(\ln \alpha + \ln x \right) \\ - \cancel{\alpha} - \cancel{(\alpha-1)x} + \cancel{\alpha x} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} \\ + \underbrace{\frac{1}{2} \ln x}_{\ln \sqrt{x}} - \underbrace{2 \ln((\alpha-1)x)}_{-\ln \sqrt{(\alpha-1)^3 x^3}} + \underbrace{\frac{1}{2} \ln((\alpha-1)x)}_{\ln \sqrt{\alpha x}} + \underbrace{\ln(\alpha x) - \frac{1}{2} \ln(\alpha x)}_{\ln \sqrt{\alpha x}} \\ = x \left((\alpha-1) \ln(\alpha-1) - \alpha \ln \alpha \right) + \ln \sqrt{\frac{\alpha}{(\alpha-1)^3 x}} + O(1)$$

taking back the exponential, we get

$$f(x) \sim \sqrt{\frac{2\pi \alpha}{(\alpha-1)^3 x}} e^{(\alpha-1) \ln(\alpha-1) - \alpha \ln \alpha} x$$

as before, but this way is actually much more involved!

Convolution of Gaussians and Green's function of a random walk

$$1. (f * g)(x) = \int_{-\infty}^{+\infty} f(t) g(x-t) dt = \int_{-\infty}^{+\infty} du f(x-u) g(u) = (g * f)(x)$$

\uparrow
 $u = x-t$

$$2. \boxed{H(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-ikx} \int_{-\infty}^{+\infty} dt f(t) g(x-t)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt f(t) \underbrace{\int_{-\infty}^{+\infty} dx e^{-ikx} g(x-t)}_{e^{-ikt} \int_{-\infty}^{+\infty} du e^{-iku} g(u)}$$

$$= \sqrt{2\pi} \left(\int_{-\infty}^{+\infty} dt f(t) e^{-ikt} \right) \left(\int_{-\infty}^{+\infty} du g(u) e^{-iku} \right) = \boxed{\sqrt{2\pi} F(k) G(k)}$$

$$3. F_{a,\sigma}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-ikx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-a)^2/2\sigma^2}$$

$$= \frac{1}{\frac{1}{\sigma} \sqrt{2\pi\sigma^2}} e^{-ika} \int_{-\infty}^{+\infty} dy e^{-iky} e^{-y^2/2\sigma^2}$$

$y = x-a$

↓ formula from the lectures
 $\sqrt{2\pi\sigma^2} e^{-k^2/2\sigma^2}$

so

$$\boxed{F_{a,\sigma}(k) = \frac{e^{-ika}}{\sqrt{2\pi}} e^{-\sigma^2 k^2/2}}$$

4. Let $h_{a,a',\sigma,\sigma'} = f_{a,\sigma} * f_{a',\sigma'}$, in Fourier space we have, from above

$$H_{a,a',\sigma,\sigma'}(k) = F_{a,\sigma}(k) F_{a',\sigma'}(k) = \frac{e^{-ik(a+a')}}{\sqrt{2\pi}} e^{-(\sigma^2 + \sigma'^2)k^2/2} = F_{a+a',\sigma+\sigma'}(k)$$

so

$$\boxed{h_{a,a',\sigma,\sigma'}(x) = f_{a+a',\sigma+\sigma'}(x)} \text{ by inverse Fourier transform.}$$

the convolution of two gaussians is a gaussian of mean $a+a'$ and variance $\sigma^2 + \sigma'^2$.

5. sketch:

$$P_{t+2}(x) = \int_{-\infty}^{+\infty} P_t(x-y) p(y) dy$$

The probability to be at x at time $t+2$ corresponds to the probability to be at $x-y$ at time t and to make a jump of y with probability $p(y)dy$. Then, sum up all contributions (or \Rightarrow "or") from all possible y .

6. we see, using 1. that

$$P_{t+z}(x) = (P_t * P)(x) = (P * P_t)(x)$$

so iteratively : $P_z = P * P_0$

$$P_{2z} = P * P * P_0$$

$$\boxed{P_{nz} = \underbrace{P * \dots * P}_n * P_0} \quad (\rightarrow \text{initial distribution})$$

7.a. To get the Green function, we have to take $P_0(x) = \delta(x)$ the Dirac distribution.

b. but $(f * \delta)(x) = (\delta * f)(x) = \int_{-\infty}^{+\infty} \delta(t) f(x-t) dt = f(x)$ shows that

$$\boxed{f * \delta = \delta * f = f} \quad \text{so here, } P * \delta = P \text{ gives}$$

$$\boxed{G_{nz}(x) = \underbrace{(P * \dots * P)}_n(x)}$$

c. applying the result of 4. it is clear that

$$G_z = f_{a,z} \quad \dots \quad G_{nz} = f_{na,nz} \quad \text{with } n = t/z$$

$$G_{2z} = f_{2a,2z}$$

so $\boxed{G_{nz}(x) = \frac{1}{\sqrt{2\pi \frac{t\sigma^2}{z}}} e^{-\frac{(x-(a/z)t)^2}{2\frac{t\sigma^2}{z}}}}$

to make connection with the usual definition of diffusion processes, we set $v=a/z$ the drift velocity and $D=\frac{\sigma^2}{2z}$ the diffusion constant so

that

$$\boxed{G(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-vt)^2}{4Dt}}}$$

Stochastic matrices:

1. we must have probability conservation:

$$\sum_{i=1}^N p_i(t+1) = 1 = \sum_i \sum_j a_{ij} p_j(t)$$

and this should be true for all stochastic vector $\vec{P}(t)$, in particular if

$p_j(t) = \delta_{jj_0}$ with given j_0 , so we must have

$$1 = \sum_i \sum_j a_{ij} \delta_{jj_0} = \sum_i a_{i,j_0}$$

sum of entries in column j_0

2. we have for a couple (\vec{v}_n, μ_n) that

$$\vec{v}_n^T A = \mu_n \vec{v}_n^T \Leftrightarrow A^T \vec{v}_n = \mu_n^* \vec{v}_n$$

$\hookrightarrow A$ is real

so \vec{v}_n is a righteigenvector of A^T with eigenvalue μ_n^* .

Since A^T and A have the same spectrum μ_n^* belongs to $\{\lambda_n\}$

Since A is real, the complex eigenvalues comes in pair μ_n and μ_n^* are both eigenvalues so finally $\{\mu_n\} = \{\lambda_n\}$.

(we forgot the μ_n notation)

3. let m be such that $|v_{m,n}| = \max_i |v_{i,n}|$, using $\vec{v}_n^T A = \lambda_n \vec{v}_n^T$, we

have $v_{m,n}^* \lambda_n = \sum_i v_{i,n}^* a_{ij} \Rightarrow |\lambda_n| |v_{m,n}| \leq \sum_i |a_{ij}| \cdot |v_{i,n}| \leq |v_{m,n}|$

\nearrow real positive

$$\leq \left(\sum_i |a_{ij}| \right) |v_{m,n}|$$

\downarrow from 1.

so finally $|\lambda_n| \leq 1$

b. similarly, we write $A \vec{u}_n = \lambda_n \vec{u}_n$ and select the m component

such that $\forall i, |\mu_{i,n}| \leq |\mu_{m,n}|$ so that $\lambda_n u_{m,n} = \sum_j a_{mj} u_{j,n}$

$$(\lambda_n - a_{mm}) u_{m,n} = \sum_{j \neq m} a_{mj} u_{j,n}$$

\uparrow from 1.

$$\text{and } |\lambda_n - a_{mm}| |\mu_{m,n}| \leq \sum_{j \neq m} a_{mj} |\mu_{j,n}| \leq \left(\sum_{j \neq m} a_{mj} \right) |\mu_{m,n}| \Rightarrow |\lambda_n - a_{mm}| \leq \sum_{j \neq m} a_{mj} = 1 - a_{mm}$$

4. we compute $(\vec{w}^+ A)_{ij} = \sum_i w_i^* a_{ij} = \frac{1}{N} \underbrace{\sum_i a_{ij}}_{=1 \text{ from 1.}} = \frac{1}{N} = w_j^*$

so we have $\boxed{\vec{w}^+ A = \vec{w}^+}$: \vec{w} is a left eigenvector of A with eigenvalue 1.

since we use ascending order for the eigenvalues $\boxed{\lambda_N = 1}$ is always present in the matrix.

5. We decompose the initial vector as:

$$\vec{P}(0) = \sum_n c_n \vec{u}_n \quad \text{with } c_n = \vec{u}_n^+ \vec{P}(0)$$

no power t , not transpose

then, $A \vec{P}(0) = \sum_n c_n \lambda_n \vec{u}_n$ and $\boxed{\vec{P}(t) = A^t \vec{P}(0) = \sum_n c_n \lambda_n^t \vec{u}_n}$

6. Let us consider that there are two eigenvalues such that $|\lambda_n| = 1$ we have necessarily $\lambda_N = 1$ and $\lambda_{N-1} = \pm 1$ (complex eigenvalues would come in pairs)

then, the eigenvalues can be written as $\lambda_n = p_n e^{i\theta_n t}$ with $p_n = |\lambda_n|$

$$\text{and } \vec{P}(t) = c_N \vec{u}_N + c_{N-1} e^{i\theta_{N-1} t} \vec{u}_{N-1} + \sum_{n < N-1} c_n p_n^t e^{i\theta_n t} \vec{u}_n$$

↳ since $p_n < 1$, they decay to zero.

$$\vec{P}(t \rightarrow \infty) = c_N \vec{u}_N + c_{N-1} e^{i\theta_{N-1} t} \vec{u}_{N-1}$$

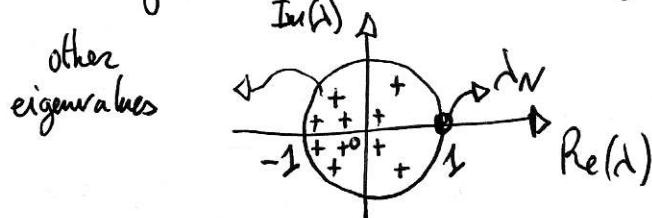
if $\theta_{N-1} = 0$: $\vec{P}(t \rightarrow \infty)$ depends on c_N and c_{N-1} , is not independent of $\vec{P}(0)$

if $\theta_{N-1} = \pi$: there is in addition some oscillations $(-1)^t$ in time

so the stationary state, independent of $\vec{P}(0)$ do not exist.

If there are complex eigenvalues, oscillations are present.

The stationary state is basically well defined if $\lambda_N = 1$ is non-degenerate and the only one with $|\lambda_n| = 1$, graphically the spectrum should look like



7. We thus have,

$$\vec{P}(t) = c_N \vec{\mu}_N + c_{N-1} p_{N-1}^t e^{i\lambda_{N-1} t} \vec{\mu}_{N-1} + \dots$$

since $p_{n < N} < 1$, $\vec{P}^\infty = c_N \vec{\mu}_N$ and since $\|\vec{\mu}_N\|^2 = 1$, $|c_N|^2 = \|\vec{P}^\infty\|^2$
 ↶ taken real
 $\therefore c_N = \|\vec{P}^\infty\|$

N.B.: $\sum_i p_i^\infty = 1$ but $\sum_i p_i^\infty 2 \leq 1$

so \vec{P}^∞ is the (unnormalized) right eigenvector of A associated to $\lambda = 1$

as expected for a stationary distribution: $[A \vec{P}^\infty = \vec{P}^\infty]$

b. it follows that, assuming λ_{N-1} non degenerate (real)

$$\|\vec{P}(t) - \vec{P}^\infty\| = |c_{N-1}| p_{N-1}^t = |c_{N-1}| e^{-t/\zeta} \text{ with } \zeta = -\frac{1}{\ln |\lambda_{N-1}|} > 0$$

c. We have the two relations:

$$\sum_i a_{ji} p_i^\infty = p_j^\infty \quad \text{and} \quad \sum_i a_{ij} = 1$$

combining them gives: $\forall j, \boxed{\sum_i a_{ij} p_j^\infty = \sum_i a_{ji} p_i^\infty}$

8. if $p_i^\infty p_{i \rightarrow j} > p_j^\infty p_{j \rightarrow i} : A_{j \rightarrow i} = 1 \text{ so } a_{ij} = p_{j \rightarrow i}$

$$A_{i \rightarrow j} = \frac{p_j^\infty p_{j \rightarrow i}}{p_i^\infty p_{i \rightarrow j}} \text{ so } a_{ji} = \frac{p_j^\infty p_{j \rightarrow i}}{p_i^\infty p_{i \rightarrow j}} \left. \right\} \boxed{a_{ij} p_j^\infty = a_{ji} p_i^\infty}$$

in the other case, one has the same reasoning.

9. a) we can use $a_{ii} = 1 - \sum_{j \neq i} a_{ji}$ (column sums up to one)

b) by choosing P , one must ensure that $a_{ii} \geq 0$!

$$\text{in } a_{ii} = 1 - P \sum_{j \neq i} \min\left(1, \frac{p_i^\infty}{p_j^\infty}\right).$$