

## Morphism from $SU(2)$ on $SO(3)$ :

We are going to show that the  $U_{\vec{n}}(\psi)$  matrices actually correspond to applying a rotation in 3D space.

To do so, we take  $\vec{x} \in \mathbb{R}^3$  and define

$$X = \vec{x} \cdot \vec{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

is an object living in  $2 \times 2$  complex matrices that represents a vector of 3D space.

We now show that the similarity transform

$$X' = U_{\vec{n}}(\psi) X U_{\vec{n}}(\psi)^+$$

is such that  $X' = \vec{x}' \cdot \vec{\sigma}$  with  $\vec{x}' = R_{\vec{n}}(\psi) \vec{x}$ ; ie the  $U_{\vec{n}}(\psi)$  matrix performed the rotation  $\vec{x} \mapsto \vec{x}'$  in 3D space!

Proof:  $X' = \left( \cos \frac{\psi}{2} - i \vec{n} \cdot \vec{\sigma} \sin \frac{\psi}{2} \right) \vec{x} \cdot \vec{\sigma} \left( \cos \frac{\psi}{2} + i \vec{n} \cdot \vec{\sigma} \sin \frac{\psi}{2} \right)$

$$= \cos^2 \frac{\psi}{2} \vec{x} \cdot \vec{\sigma} + \sin^2 \frac{\psi}{2} (\vec{n} \cdot \vec{\sigma})(\vec{x} \cdot \vec{\sigma})(\vec{n} \cdot \vec{\sigma}) + i \cos \frac{\psi}{2} \sin \frac{\psi}{2} \underbrace{[(\vec{x} \cdot \vec{\sigma})(\vec{n} \cdot \vec{\sigma})]}_{- (\vec{n} \cdot \vec{\sigma})(\vec{x} \cdot \vec{\sigma})} + \underbrace{2i(\vec{x} \wedge \vec{n}) \cdot \vec{\sigma}}$$

We are going to use extensively

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b}) \mathbb{1} + i(\vec{a} \wedge \vec{b}) \cdot \vec{\sigma}$$

$$\begin{aligned} (\vec{n} \cdot \vec{\sigma})(\vec{x} \cdot \vec{\sigma})(\vec{n} \cdot \vec{\sigma}) &= (\vec{n} \cdot \vec{\sigma}) \left( (\vec{x} \cdot \vec{n}) \mathbb{1} + i(\vec{x} \wedge \vec{n}) \cdot \vec{\sigma} \right) \\ &= ((\vec{x} \cdot \vec{n}) \vec{n}) \cdot \vec{\sigma} + i \left( \vec{n} \cdot (\vec{x} \wedge \vec{n}) \mathbb{1} + i \underbrace{\vec{n} \wedge (\vec{x} \wedge \vec{n})}_{\vec{n}^2 \vec{\sigma} - (\vec{x} \cdot \vec{n}) \vec{n}} \cdot \vec{\sigma} \right) \\ &= ((\vec{x} \cdot \vec{n}) \vec{n} - \vec{x}) \cdot \vec{\sigma} \end{aligned}$$

so  $X' = \left\{ \underbrace{\left( \cos^2 \frac{\psi}{2} - \sin^2 \frac{\psi}{2} \right)}_{\cos \psi} \vec{x} + \underbrace{2 \sin^2 \frac{\psi}{2} (\vec{x} \cdot \vec{n}) \vec{n}}_{1 - \cos \psi} + \sin \psi (\vec{n} \wedge \vec{x}) \right\} \cdot \vec{\sigma}$

$$= \vec{x}' \cdot \vec{\sigma} \text{ with } \vec{x}' = R_{\vec{n}}(\psi) \vec{x} \text{ from Rodrigues formula.}$$