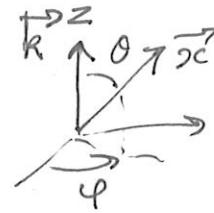


Isotropy

$$F(\vec{k}) = \int_{\mathbb{R}^3} d\vec{x} f(\vec{x}) e^{-i\vec{k} \cdot \vec{x}}$$



take z
along \vec{k}
 $k=|\vec{k}|$

in spherical coordinates: $(x, y, z) \rightarrow (r, \theta, \varphi)$

$$d\vec{x} \rightarrow r^2 \sin \theta d\theta d\varphi dr$$

$$= \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi r^2 \sin \theta e^{-ikr \cos \theta} f(r)$$

$$= 2\pi \int_0^\infty dr r^2 f(r) \underbrace{\int_0^\pi d\theta \sin \theta e^{-ikr \cos \theta}}_{\left[-\frac{e^{-ikr \cos \theta}}{ikr} \right]_0^\pi} = 2 \frac{\sin(kr)}{kr}$$

$$F(\vec{k}) = \frac{4\pi}{k} \int_0^\infty dr r \sin(kr) f(r)$$

is isotropic in \vec{k} -space!

Correlator:

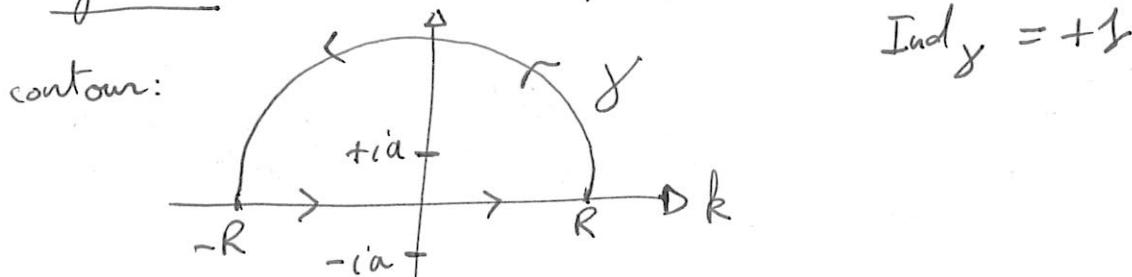
1. $d=1$: $f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{k^2 + a^2}$

two poles for $f(z) = \frac{e^{izx}}{z^2 + a^2} \cdot \frac{1}{2\pi}$
 $\hookrightarrow z = \pm ia$

using residue theorem, we need to have

$|e^{izx}| \rightarrow 0$ when $z \rightarrow \infty$

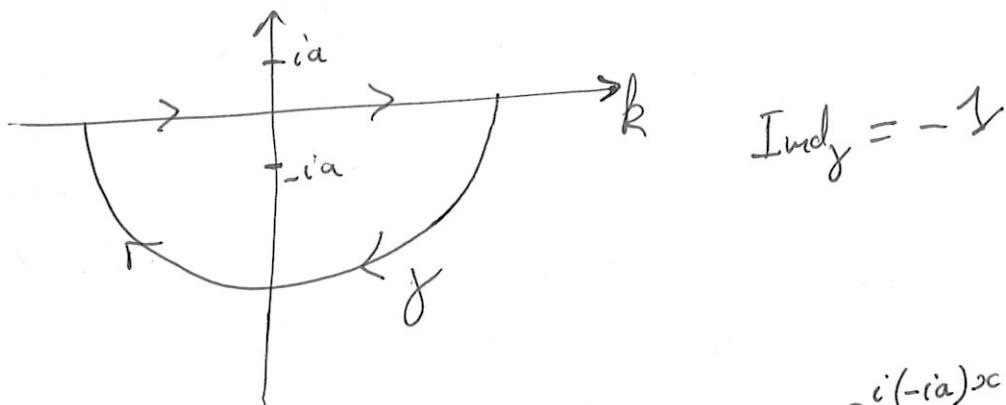
• if $x > 0$: choose $\text{Im}(z) > 0$, $e^{-\text{Im}(z)x} \rightarrow 0$



$$\oint \frac{f(z) dz}{z} = 2i\pi \text{Res}(ia) = \frac{2i\pi}{2\pi} \frac{e^{i(ia)x}}{2(ia)} = \frac{e^{-ax}}{2a}$$

$$\gamma \quad "I_R + I_{\text{circle}}" = \int_{-\infty}^{+\infty} dk f(k)$$

if $x < 0$: then $\operatorname{Im}(z) < 0$



residue: $\oint_\gamma f(z) dz = -2i\pi \operatorname{Res}(-ia) = -\frac{2i\pi}{2\pi} \frac{e^{i(-ia)x}}{2(-ia)} = \frac{e^{ax}}{2a}$

$\uparrow \quad \operatorname{Ind}_\gamma$

$= \frac{e^{-ax}}{2a}$

finally

$$\boxed{f(x) = \frac{e^{-ax}}{2a}}$$

$d=3$: in spherical coordinates in \vec{k} -space

$$f(\vec{x}) = \frac{1}{(2\pi)^3} \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi \ k^2 \sin\theta \frac{e^{ikx \cos\theta}}{\vec{k}^2 + a^2}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{\vec{k}^2 + a^2} \left[\frac{e^{ikx \cos\theta}}{ikx} \right]_0^\pi = \frac{2}{4\pi^2} \int_0^\infty dk \frac{k}{\vec{k}^2 + a^2} \frac{\sin(kx)}{kx}$$

$\hookrightarrow 2 \frac{\sin(kx)}{kx}$

and

$$= \frac{1}{4\pi^2 x} \operatorname{Im} \left(\int_{-\infty}^{+\infty} dk \frac{ke^{ikx}}{\vec{k}^2 + a^2} \right)$$

$\hookrightarrow \begin{cases} \text{as } x > 0 \\ = 2i\pi \end{cases} = \frac{(ia)e^{i(-ia)x}}{2ia} = 2\pi e^{-ax}$

$$\boxed{f(\vec{x}) = \frac{e^{-ax}}{4\pi x}}$$

2. taking $a \rightarrow 0$, we have $\frac{1}{4\pi x} = \int_{R^3} \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k} \cdot \vec{x}}}{\vec{k}^2}$ so $\operatorname{TF}\left(\frac{1}{4\pi x}\right) = \frac{4\pi}{R^2}$

3. we have

$$\operatorname{TF}\left(\frac{1}{x^2 + a^2}\right) = 2\pi \frac{e^{-ax}}{2a}$$

and using $\operatorname{T.F.}(f'(x)) = ik \operatorname{T.F.}(f(x))$
with here $f'(x) = \frac{-2x}{(x^2 + a^2)^2}$

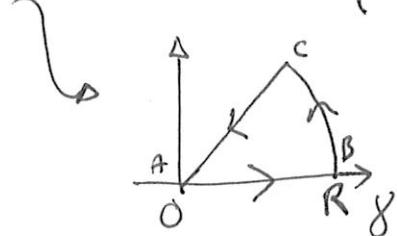
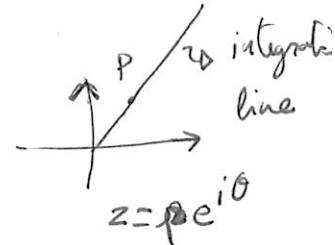
one obtains

$$\begin{aligned} \text{TF}\left(\frac{\omega}{(x^2+a^2)^2}\right) &= -\bar{z}^3 ik \text{TF}\left(\frac{1}{(x^2+a^2)}\right) \\ &= -\frac{2\pi ik}{z} \frac{e^{-|k|l}}{2a} = -i\pi k \frac{e^{-|k|l}}{2a} \end{aligned}$$

Power-Law

Laplace transform: $\mathcal{L}(p) = \int_0^\infty dx f(x) e^{-px}$

$$\begin{aligned} 1. \quad \mathcal{L}(p) &= \int_0^\infty dx x^s e^{-px} \\ &= \frac{1}{p^{s+1}} \int_{|z|=0}^\infty dz z^{s+1-1} e^{-z} \\ &\quad \uparrow z=px \\ &= \frac{\Gamma(s+1)}{p^{s+1}} \end{aligned}$$



$$\text{so } \boxed{\mathcal{L}(p) = \frac{\Gamma(s+1)}{p^{s+1}}} \quad \text{Re } p > 0 \quad (\Rightarrow \text{includes T.F. with } p=ik)$$

$$\oint_C z^s e^{-z} dz = 0$$

$$(\Rightarrow \int_0^R dx x^s e^{-px} + I_{BC})$$

$$+ - \int_0^R d\rho \rho^s e^{i\theta s} e^{-\rho e^{i\theta}} = 0$$

$$(\Rightarrow \text{whatever } \theta \rightarrow \Gamma(s+1))$$

2. let $p=ik$

$$\mathcal{L}(ik) = \int_0^\infty dx f(x) e^{-ikx}$$

~~REMARK~~

$$= \int_{-\infty}^{+\infty} dx g(x) e^{-ikx}$$

$$= \int_0^{+\infty} dx g(x) e^{-ikx} + \int_{-\infty}^0 dx g(x) e^{-ikx}$$

$$= \int_0^\infty dx f(x) e^{-ikx} + \int_0^\infty dx g(-x) e^{ikx}$$

$$= \mathcal{L}(ik) + \mathcal{L}(-ik) = \int_0^\infty dx f(x) 2\cos(kx) = \boxed{2\operatorname{Re}(\mathcal{L}(ik))}$$

$$3. \quad g(x) = 1/|x|^{\alpha} \quad ; \quad s = -\alpha > -1 \Rightarrow \alpha < 1; \quad ik = |k| e^{i\frac{\pi}{2}} \\ \text{sign}(k)$$

$$\text{T.F.}(g) = 2 \operatorname{Re} \left(\frac{\Gamma(\alpha+1)}{(ik)^{\alpha+1}} \right) = 2 \frac{\Gamma(1-\alpha)}{|k|^{1-\alpha}} \underbrace{\operatorname{Re} \left(e^{i \operatorname{sign}(k) \frac{\pi i}{2} (\alpha-1)} \right)}_{\cos \left(\frac{\pi}{2} (\alpha-1) \right) = \cos \left(\frac{\pi}{2} - \frac{\pi}{2} \alpha \right)} \\ = \sin \left(\frac{\pi}{2} \alpha \right)$$

$$\boxed{\text{T.F.} \left(\frac{1}{|p|^{\alpha}} \right) = 2 \frac{\Gamma(1-\alpha)}{|k|^{1-\alpha}} \sin \left(\frac{\pi}{2} \alpha \right)}$$