

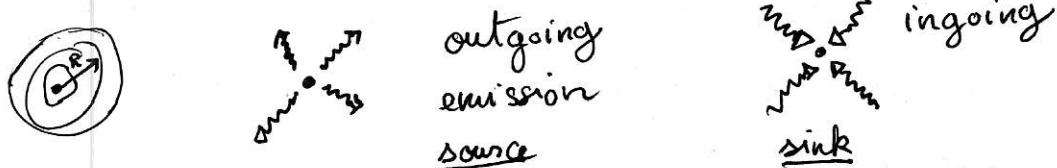
b) Three dimensional case :

This is the usual one and the discussion is interesting because it shows some slight differences although the ideas are essentially the same.

□ Spherical symmetry:

we have $\left[\vec{\nabla}^2 - \frac{\partial^2}{\partial t^2} \right] G(\vec{r}, \vec{r}', z) = \delta(\vec{r} - \vec{r}') \delta(z)$, $z = t - t'$

and infinite system. Rotational invariance suggest to switch to spherical coordinates centered on \vec{r}' and from radius $R = |\vec{r} - \vec{r}'|$



in what follows, we are looking for outgoing solutions only, with the causality conditions $R \ll cZ$.

The change of variables $\vec{r} - \vec{r}' \rightarrow (R, \theta, \phi)$ leads to $\delta(\vec{r} - \vec{r}') = \frac{1}{R^2 \sin \theta} \delta(R) \delta(\theta) \delta(\phi)$

$$d(\vec{r} - \vec{r}') \rightarrow R^2 dR \underbrace{\sin \theta d\theta d\phi}_{d\Omega} \rightarrow 4\pi R^2 dR$$

↑ integrating the angles

we assume $G(\vec{r}, \vec{r}', z) = G(R, z)$ then, the action of the Laplacian reads:

$$\hat{\Delta}_{\vec{r}} G = \frac{1}{R^2} \left(\frac{\partial}{\partial R} \left(R^2 \frac{\partial G}{\partial R} \right) \right) \text{ so that the equation reads for } \tilde{G}(R, \omega):$$

$$\frac{1}{R^2} \left(\frac{\partial}{\partial R} \left(R^2 \frac{\partial \tilde{G}}{\partial R} \right) \right) + \omega^2 \tilde{G} = \frac{1}{R^2 \sin \theta} \delta(R) \delta(\theta) \delta(\phi)$$

Let us integrate over the angles $\int_{4\pi} d\Omega$ the equation and multiply by R^2 :

$$4\pi \left(\frac{\partial}{\partial R} \left(R^2 \frac{\partial \tilde{G}}{\partial R} \right) + R^2 \omega^2 \tilde{G} \right) = \delta(R)$$

Then, as for the static case, we can look for a solution of the form $\tilde{G}(R, \omega) = \frac{f(R, \omega)}{4\pi R}$; $\frac{\partial \tilde{G}}{\partial R} = \frac{1}{4\pi} \left(-\frac{f'}{R^2} + \frac{f}{R} \right)$; $f' = \frac{\partial f}{\partial R}$

$$\text{and } \frac{\partial}{\partial R} \left(R^2 \frac{\partial \tilde{G}}{\partial R} \right) = \frac{1}{4\pi} \frac{\partial}{\partial R} (-f + Rf') = \frac{1}{4\pi} (-f' + f' + Rf'')$$

so the equation for f reads:

$$R \left(\frac{\partial^2 f}{\partial R^2} + \omega^2 f \right) = S(R)$$

For $R \neq 0$, the solutions of the equation is

$$f(R, \omega) = A e^{i\omega R} + B e^{-i\omega R}$$

\hookrightarrow outgoing \hookrightarrow ingoing $\Rightarrow B=0$

Now, we must set A according to the boundary conditions. The condition is given by the integration of the S pulse at $R=0$ over a small sphere of radius ϵ :

we have $\int_{S_\epsilon} d\vec{R} G d\epsilon^2 \rightarrow 0$ if f finite at $R \rightarrow 0$ (which is the case...)

$$\int_{S_\epsilon} \Delta \tilde{G}(\vec{R}, \omega) d\vec{R} = \int_{\partial S_\epsilon} \vec{\nabla} \tilde{G} \cdot d\vec{S} = 4\pi \epsilon^2 \frac{\partial \tilde{G}}{\partial R} \Big|_{R=\epsilon}$$

Green's theorem

$$\int_{S_\epsilon} \delta(\vec{r} - \vec{r}') d\vec{R} = 1$$

$$\frac{\partial \tilde{G}}{\partial R} \Big|_{R=\epsilon} = \frac{1}{4\pi \epsilon^2}$$

$$\text{but } \frac{\partial \tilde{G}}{\partial R} \Big|_{R=\epsilon} = \frac{1}{4\pi} \left(-\frac{f}{\epsilon^2} + \frac{f'}{\epsilon} \right) \Rightarrow -f(\epsilon, \omega) + \epsilon f'(\epsilon, \omega) \xrightarrow{\substack{\parallel \\ \parallel \\ A \\ i\omega A}} 1 \Rightarrow A = -1$$

Finally,

$$\tilde{G}(R, \omega) = -\frac{e^{i\omega R}}{4\pi R} \quad \text{no poles in } \omega !$$

□ Last step is inverse Fourier transform: we get a delta instead of Θ

$$G^n(\vec{r} - \vec{r}', z) = -\frac{1}{4\pi R} \underbrace{\int_{-\infty}^{+\infty} \frac{dw}{2\pi} e^{-i\omega(z-R)}}_{\delta(z-R)} = -\frac{\delta(t-t'-||\vec{r}-\vec{r}'||)}{4\pi ||\vec{r}-\vec{r}'||}$$

This is the so-called retarded potential formula in electromagnetism:

$$\left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) V(\vec{r}, t) = -\frac{P(\vec{r}, t)}{\epsilon_0} \Rightarrow$$

Δ potential

$$V(\vec{r}, t) = \frac{1}{4\pi \epsilon_0} \int d\vec{r}' \frac{P(\vec{r}', t - ||\vec{r}-\vec{r}'||/c)}{||\vec{r}-\vec{r}'||}$$

charges