

④ Laplace's method

with the following assumptions

$$f(x) = \int_{t_c}^b dt g(t) e^{-xh(t)}$$

with, as $t \rightarrow t_c^+$

$$h(t) \approx h_c + \sum_{n=0}^{\infty} a_n (t-t_c)^{\mu+n}, \mu > 0, a_0 \neq 0$$

$$g(t) \approx \sum_{n=0}^{\infty} b_n (t-t_c)^{\beta-1+n}, \beta > 0, b_0 \neq 0$$

so we have

$$h'(t) = \sum_{n=0}^{\infty} a_n (\mu+n)(t-t_c)^{\mu-1+n}$$

Now, there exists $c \in [t_c, b]$ such that $h'(t) > 0$ on $[t_c, c]$, we set

$U = h(c) - h_c$ and make the change of variables $u = h(t) - h_c > 0$

in the following leading contribution:

$$\boxed{f(x) = \int_{t_c}^c dt g(t) e^{-xh(t)} = e^{-xc h_c} \int_0^u du \frac{g(t(u))}{h'(t(u))} e^{-x c u}}$$

kill contribution above $u^{1/\mu}$

The difficulty lies in the $\frac{g}{h'}$ prefactor and the $t(u) = h^{-1}(u+h_c)$ inversion. This is performed by considering the first terms of the expansions.

We expect that

$$u = \sum_{n=0}^{\infty} a_n (t-t_c)^{\mu+n} \quad (i) \text{ can be inverted into}$$

$$t-t_c = \sum_{s=1}^{\infty} \alpha_s u^{s/\mu} \quad (ii)$$

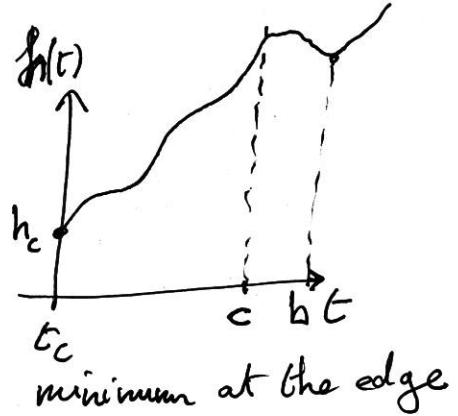
the coefficients α_s are determined by injecting (ii) back in (i)

$$\text{so that: } u = a_0 (\alpha_1 u^{1/\mu} + \alpha_2 u^{2/\mu} + \alpha_3 u^{3/\mu} + \dots)^{\mu} \\ + a_1 (\alpha_1 u^{1/\mu} + \alpha_2 u^{2/\mu} + \alpha_3 u^{3/\mu} + \dots)^{\mu+1} \\ + a_2 (\alpha_1 u^{1/\mu} + \alpha_2 u^{2/\mu} + \alpha_3 u^{3/\mu} + \dots)^{\mu+2} + \dots$$

(first three terms)

identifying terms in u :

$$1 = a_0 \alpha_1^{\mu} \Rightarrow \boxed{\alpha_1 = 1/a_0^{1/\mu}}$$



we may then identify terms in $\mu^{1+\frac{1}{\mu}}$ and $\mu^{1+\frac{2}{\mu}}$ so we expand the powers:

$$\begin{aligned} \mu &= a_0 \alpha_1^\mu \mu \left(1 + \frac{\alpha_2}{\alpha_1} \mu^{1/\mu} + \frac{\alpha_3}{\alpha_1} \mu^{2/\mu} + \dots\right)^\mu \\ &\quad + a_1 \alpha_1^{\mu+1} \mu^{1+\frac{1}{\mu}} \left(1 + \frac{\alpha_2}{\alpha_1} \mu^{1/\mu} + \dots\right)^{\mu+1} \\ &\quad + a_2 \alpha_1^{\mu+2} \mu^{1+\frac{2}{\mu}} \\ &= a_0 \alpha_1^\mu \mu \left(1 + \mu \frac{\alpha_2}{\alpha_1} \mu^{1/\mu} + \left(\frac{\alpha_3}{\alpha_1} \mu + \frac{\mu(\mu-1)}{2} \frac{\alpha_2}{\alpha_1}\right) \mu^{2/\mu} + \dots\right) \\ &\quad + a_1 \alpha_1^{\mu+1} \mu^{1+\frac{1}{\mu}} \left(1 + (\mu+1) \frac{\alpha_2}{\alpha_1} \mu^{1/\mu} + \dots\right) \\ &\quad + a_2 \alpha_1^{\mu+2} \mu^{1+\frac{2}{\mu}} \end{aligned}$$

Identifying power μ : $\boxed{\alpha_1 = a_0^{-1/\mu}}$ (again ...)

$$\begin{array}{c} \text{---} \\ \text{"} \quad \text{"} \quad \mu^{1+\frac{1}{\mu}}: \\ \text{---} \\ \text{"} \quad \text{"} \quad \mu^{1+\frac{2}{\mu}}: \end{array} 0 = a_0 \alpha_1^\mu \mu \frac{\alpha_2}{\alpha_1} + a_1 \alpha_1^{\mu+1} \Rightarrow \boxed{\alpha_2 = -\frac{\alpha_1}{\mu a_0^{1+2/\mu}}}$$

$$0 = a_0 \alpha_1^\mu \left(\mu \frac{\alpha_3}{\alpha_1} + \frac{\mu(\mu-1)}{2} \frac{\alpha_2}{\alpha_1} \right) + a_1 \alpha_1^{\mu+1} (\mu+1) \frac{\alpha_2}{\alpha_1} + a_2 \alpha_1^{\mu+2}$$

after some simplifications, one gets

$$\boxed{\alpha_3 = \frac{(\mu+3)\alpha_1^2 - 2\mu a_0 \alpha_2}{2\mu^2 a_0^{2+3/\mu}}}$$

then, one must get the expansion of $\frac{g(t(\mu))}{h'(t(\mu))}$ as a function of μ and
(here starts the nightmare, we only show the leading term):

$$\begin{aligned} g(t(\mu)) &\simeq b_0 (\alpha_1 \mu^{1/\mu})^{\beta-1} \simeq b_0 \alpha_1^{\beta-1} \mu^{(\beta-1)/\mu} \\ h'(t(\mu)) &\simeq a_0 \mu (\alpha_1 \mu^{1/\mu})^{\mu-1} \simeq a_0 \mu \alpha_1^{\mu-1} \mu^{1-1/\mu} \\ \text{so } \tilde{f}(x) &\simeq e^{-\omega c h_c} \int_0^T du c_0 \mu^{1/\mu-1} e^{-\omega c u} = c_0 e^{-\omega c h_c} \frac{1}{\omega c^{1/\mu}} \int_0^{T x} dv v^{\frac{1}{\mu}-1} e^{-v} \\ &\qquad \qquad \qquad \uparrow \qquad \qquad \qquad \underbrace{\qquad \qquad \qquad}_{v=\mu x} \qquad \qquad \qquad \approx \Gamma'(\beta/\mu) \end{aligned}$$

$$\text{with } c_0 = \frac{b_0}{a_0 \mu} \alpha_1^{\beta-\mu}$$

$$\boxed{c_0 = \frac{b_0}{\mu a_0^{\beta-1/\mu}}}$$

the other terms are obtained by looking at the expansion (horrible ...)

suppose now that we were able to get the c_n coefficients, then we have the following expansion

$$\frac{g(t(u))}{h'(t(u))} = \sum_{n=0}^{\infty} c_n u^{\frac{\beta+n}{\mu}-1}$$

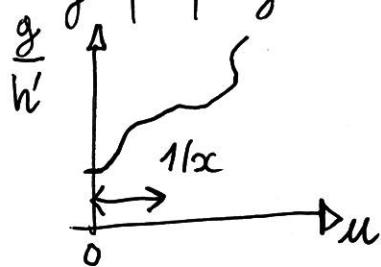
which we inject in the integral (inverting the sum and the series under the asymptotic meaning n)

$$\begin{aligned}\tilde{f}(x) &\sim e^{-xh_c} \sum_{n=0}^{\infty} c_n \int_0^U du u^{\frac{\beta+n}{\mu}-1} e^{-xu} \\ &= e^{-xh_c} \sum_{n=0}^{\infty} c_n \frac{1}{x^{\frac{\beta+n}{\mu}}} \underbrace{\int_0^{Ux} dv v^{\frac{\beta+n}{\mu}-1} e^{-v}}_{\simeq \Gamma\left(\frac{\beta+n}{\mu}\right)}\end{aligned}$$

hence the following (almost explicit expansion):

$$\boxed{\tilde{f}(x) \sim e^{-xh_c} \sum_{n=0}^{\infty} \Gamma\left(\frac{\beta+n}{\mu}\right) \frac{c_n}{x^{(\beta+n)/\mu}}}$$

graphically, the e^{-ux} term picks up the contributions from the local expansion of $\frac{g}{h'}$ over a distance $1/x$. The difficulty lies in getting properly this expansion, i.e. the c_n coefficients.



Last, the contribution of the rest of the integral $\int_c^b dt g(t) e^{-xh(t)}$ is negligible with respect to the powers in the expansion.

if $G = \max_{[c,b]} |g(t)|$, $h_m = \min_{[c,b]} h(t)$ then

$$\left| e^{xh_c} \int_c^b dt g(t) e^{-xh(t)} \right| \leq \int_c^b dt |g(t)| e^{-x(h(t)-h_c)} \leq G e^{-x(h_m-h_c)} \text{ goes exponentially fast to zero.}$$

We can check that the Laplace method gives back the steepest descent result in a particular case.

for $f(x) = \int_{t_c}^b dt e^{-xh(t)}$ with $h(t)$ analytic at t_c

gives $g(t) = 1 \Rightarrow b_0 = 1; b_{i>0} = 0; \beta = 1$

$$h(t) \approx h_c + \frac{h_c''}{2}(t-t_c)^2 \Rightarrow \mu = 2; a_0 = \frac{h_c''}{2}$$

thus $c_0 = \frac{1}{2\left(\frac{h_c''}{2}\right)^{1/2}}$ and $\Gamma\left(\frac{1}{\mu}\right) = \Gamma(1/2) = \sqrt{\pi}$

so that $f(x) \sim \frac{1}{2} \sqrt{\frac{2\pi}{h_c''}} \frac{e^{-xh_c}}{\Gamma(1/2)}$ which is half the contribution of the steepest-descent because the gaussian is cut into two.

Application to the corrections to Stirling formula:

first rewrite the Gamma function as

$$\Gamma(x+1) = \int_0^{+\infty} dv v^x e^{-v} = \left(\frac{x}{e}\right)^x \int_{-1}^{+\infty} dt e^{-xh(t)}$$

\uparrow
 $t = x(1+v)$

with $h(t) = t - \ln(1+t)$

$h'(t) = 1 - \frac{1}{1+t}$ has a minimum at $t=0$

then treat separately $\int_0^{+\infty} dt e^{-xh(t)}$ and $\int_{-1}^0 dt e^{-xh(t)} = \int_0^1 dt e^{-xh(-t)}$

with Laplace method.

Using $h(t) \approx \frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{4}$ we have $a_0 = \frac{1}{2}; a_1 = -\frac{1}{3}; a_2 = \frac{1}{4}; \mu = 2$

and $g(t)=1 \Rightarrow b_0 = 1; b_{i>0} = 0; \beta = 1$ and similarly for $h(t)$

you get that the $\frac{1}{t^2}$ terms cancels and that the $\frac{1}{x^2}$ term has

prefactor $\frac{1}{12}$ in the series expansion $\Gamma(x+1) \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left\{ 1 + \frac{1}{12x} + \dots \right\}$