

The Darboux - Christoffel equality:

we write

$$x P_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + C_k P_{k-1}(x)$$

$$y P_k(y) = A_k P_{k+1}(y) + B_k P_k(y) + C_k P_{k-1}(y)$$

$$\text{with } C_k = \frac{N_k}{N_{k-1}} A_{k-1}$$

$$\Rightarrow (x-y) P_k(x) P_k(y) = A_k P_{k+1}(x) P_k(y) + B_k P_k(x) P_k(y) + C_k P_{k-1}(x) P_k(y) \\ - A_k P_{k+1}(y) P_k(x) - B_k P_k(y) P_k(x) - C_k P_{k-1}(y) P_k(x)$$

$$\Rightarrow (x-y) \frac{P_k(x) P_k(y)}{N_k} = \frac{A_k}{N_k} (P_{k+1}(x) P_k(y) - P_{k+1}(y) P_k(x)) - \frac{A_{k-1}}{N_{k-1}} (P_k(x) P_{k-1}(y) - P_k(y) P_{k-1}(x))$$

so there is a cancellation term by term:

$$(x-y) \sum_{k=1}^n \frac{P_k(x) P_k(y)}{N_k} = \underbrace{\left(\frac{A_n}{N_n} (P_{n+1}(x) P_n(y) - P_{n+1}(y) P_n(x)) - \frac{A_0}{N_0} (P_1(x) P_0(y) - P_1(y) P_0(x)) \right)}_{\downarrow} \\ = \frac{a_n^0}{a_1^1} \frac{1}{N_0} (P_0(a_1^1 x + a_0^1) - P_0(a_1^1 y + a_0^1))$$

and $\left. \begin{array}{l} P_1(x) = a_1^1 x + a_0^1 \\ P_0(x) = a_0^0 \\ A_0 = \frac{a_0^0}{a_1^1} \end{array} \right\}$

$$= \frac{P_0^2}{N_0} (x-y) \rightsquigarrow \text{can be put on the left as } k=0 \text{ term}$$

Finally

$$\boxed{\sum_{k=0}^n \frac{P_k(x) P_k(y)}{N_k} = \frac{a_n^0}{N_0 a_{n+1}^{n+1}} (P_{n+1}(x) P_n(y) - P_{n+1}(y) P_n(x))}$$

$$\text{Legendre equation: } (1-x^2)y'' - 2xy' + \lambda y = 0$$

Rodrigues: $P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1-x^2)^n \stackrel{\text{Complex analysis Tutorial}}{\Rightarrow} \frac{1}{2^n} \frac{1}{2i\pi} \oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}} dz$

Weight function: $\int^x \frac{q}{p} = \int^x \frac{-2x'}{1-x'^2} dx' = \ln(1-x^2)$

$$\Rightarrow w(x) = \frac{1}{1-x^2} e^{\ln(1-x^2)} = 1 !$$

Eigenvalues: $\lambda_n = (1+n)n$

Generating function: $G(x, t) = \sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{2i\pi} \oint_C \left(\sum_{n=0}^{\infty} \left(\frac{(z^2-1)t}{2(z-x)} \right)^n \right) dz$

We assume that C is such that $\left| \frac{(z^2-1)t}{2(z-x)} \right| < 1$ to sum up the geometric series ("t small enough") gives

$$\begin{aligned} \frac{1}{1 - \frac{(z^2-1)t}{2(z-x)}} \frac{1}{z-x} &= \frac{1}{z-x - \frac{1}{2}(z^2-1)t} \\ &= -\frac{2}{t} \frac{1}{z^2 - \frac{2}{t}z + \frac{2xt-t^2}{t}} \end{aligned}$$

and $G(x, t) = -\frac{2}{t} \frac{1}{2i\pi} \oint_C \frac{dz}{(z-z_1)(z-z_2)}$ where $z_{1,2} = \frac{1}{t} \mp \sqrt{\frac{1-2xt+t^2}{t^2}}$

We consider $C = \{ |z|=1 \}$ and $t \rightarrow 0$, $z_2 \approx \frac{2}{t} \rightarrow \infty$ while $z_1 \approx \frac{1}{t} \mp \sqrt{\frac{1-2xt+t^2}{t^2}} \approx \epsilon [0, 1]$
so only z_1 must be kept in the residue calculation:

$$G(x, t) = -\frac{2}{t} \times \frac{1}{z_1 - z_2} = -\frac{2}{t} \frac{-2t}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1-2xt+t^2}}$$

Generating function for the Hermite polynomials:

$$G(x, t) = e^{-t^2 + 2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

• $\frac{\partial G}{\partial t} = (-2t + 2x) G(x, t) = \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{(n-1)!}$

$\downarrow h' = n-1$

$$\Rightarrow -2 \underbrace{\sum_{n=0}^{\infty} H_n \frac{t^{n+1}}{n!}}_{n'=n+1} + 2 \sum_{n=0}^{\infty} x H_n t^n = \sum_{n'=0}^{\infty} H_{n'+1} \frac{t^{n'}}{n'!}$$

$n' = n+1 \quad \sum_{n'=1}^{\infty} H_{n'-1} \frac{t^{n'}}{n'!} (n')$ identification of series leads to

$$\boxed{-2n H_{n-1} + 2x H_n = H_{n+1}}$$

and $H_0 = 1 ; H_1 = 2x H_0 = 2x$ etc ...

• interestingly, we can consider instead:

$$\begin{aligned} \frac{\partial G}{\partial x} &= 2t G(x, t) = 2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} 2n H_{n-1}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} H'_n(x) \frac{t^n}{n!} \end{aligned}$$

\Rightarrow gives $\boxed{H'_n(x) = 2n H_{n-1}(x)}$

the roots of H_{n-1} are the min/max of H_n and H_n has zeros between those of H_{n-1}

$$\begin{array}{ccccccccc} & 1 & 1 & 1 & 1 & 1 & & n-1 \\ 1 & & 1 & 1 & 1 & 1 & 1 & n \end{array}$$