

Integrals using Cauchy's formula

$$1. z = e^{i\theta}, dz = izd\theta \text{ or } d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}(z + \frac{1}{z}) \Rightarrow \cos^{2n}\theta = \frac{1}{2^{2n}}(z + \frac{1}{z})^{2n} = \frac{1}{2^{2n}} \sum_{m=0}^{2n} \binom{2n}{m} z^m (\frac{1}{z})^{2n-m}$$

$$\begin{aligned} \text{so } \int_0^{2\pi} \cos^{2n}\theta d\theta &= \frac{1}{2^{2n}} \sum_{|z|=1, \text{ int}} (z + \frac{1}{z})^{2n} \frac{dz}{iz} = \frac{1}{2^{2n} i} \sum_{m=0}^{2n} \left(\sum_{|z|=1, \text{ int}} z^{2m-2n-1} dz \right) \binom{2n}{m} \\ &= \frac{1}{2^{2n} i} 2\pi i \binom{2n}{n} \underbrace{\frac{(2n)!}{2^{2n} (n!)^2}}_{\text{so } m=n} \end{aligned}$$

$$2. I = \int_C \frac{e^{2z}}{(1+z)^4} dz \text{ looks like } f'''(\gamma) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-\gamma)^4} dz$$

so we take $f(z) = e^{2z}$, $f'''(z) = 8e^{2z}$ and $\gamma = -1$, this gives

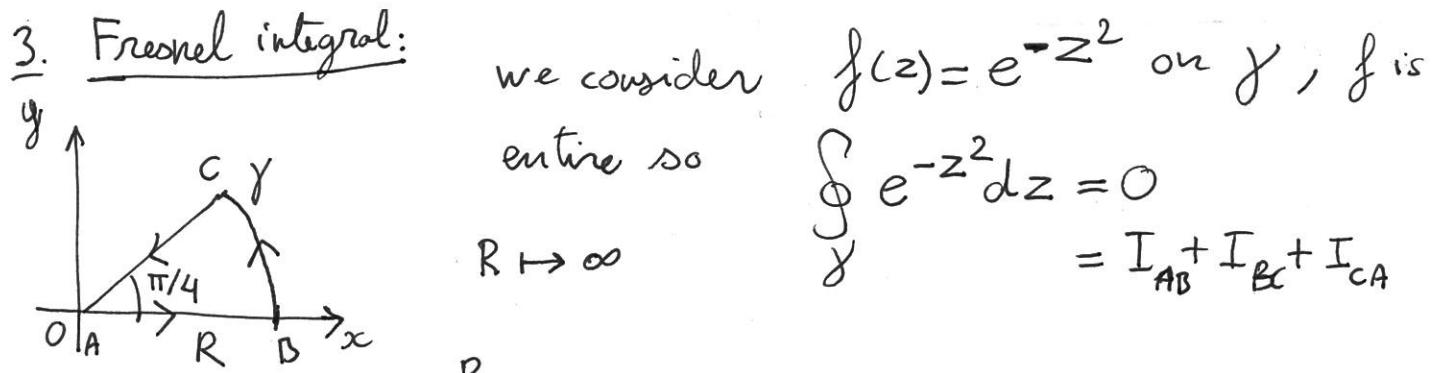
$$8e^{-2} = \frac{3!}{2\pi i} \int_C \frac{e^{2z}}{(z+1)^4} dz \Rightarrow I = \boxed{\frac{8\pi i}{3} e^{-2}}$$

$F(t) = \int_C \frac{e^{zt}}{(z^2+1)^2} dz$; we take $f(z) = e^{zt}$ in Cauchy but we need to put apart the fraction:

$$\begin{aligned} \frac{1}{(z^2+1)^2} &= \frac{1}{[(z-i)(z+i)]^2} \quad \text{and} \quad \frac{1}{(z-i)(z+i)} = \frac{z+i-(z-i)}{2i(z-i)(z+i)} = \frac{1}{2i} \left[\frac{1}{z-i} - \frac{1}{z+i} \right] \\ &= \frac{1}{-4} \left(\frac{1}{z-i} - \frac{1}{z+i} \right)^2 = -\frac{1}{4} \left(\frac{1}{(z-i)^2} + \frac{1}{(z+i)^2} - 2 \frac{1}{(z-i)(z+i)} \right) \\ &= -\frac{1}{4} \left(\frac{1}{(z-i)^2} + \frac{1}{(z+i)^2} - \frac{1}{i} \left(\frac{1}{z-i} - \frac{1}{z+i} \right) \right) \end{aligned}$$

Cauchy gives $\int_C \frac{f(z) dz}{(z-\gamma)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(\gamma)$ (same as residue theorem in this simpler case)

$$\text{so } F(t) = -\frac{1}{4} \left[\frac{1}{2!} \underbrace{te^{it} + te^{-it}}_{2t \cos t} - \frac{1}{i} \underbrace{\left(e^{it} - e^{-it} \right)}_{2i \sin t} \right] = \boxed{\frac{1}{2} [2\sin t - t \cos t]}$$



we have $I_{AB} = \int_0^R e^{-x^2} dx \xrightarrow[R \rightarrow \infty]{} \sqrt{\pi}/2$

$$\bullet I_{BC} = iR \int_0^{\pi/4} e^{i\theta} e^{-R^2 e^{2i\theta}} d\theta$$

$$z = Re^{i\theta}$$

$$dz = Rie^{i\theta} d\theta$$

$$= iz d\theta$$

we can show that $I_{BC} \rightarrow 0$ as $R \rightarrow \infty$:

$$|I_{BC}| \leq \int_0^{\pi/4} |iRe^{i\theta} e^{-R^2 e^{2i\theta}}| d\theta = R \int_0^{\pi/4} e^{-R^2 \cos(2\theta)} d\theta$$

$$R \sqrt{\frac{e^{-R^2(e^{2i\theta}-e^{-2i\theta})}}{2\cos 2\theta}} = R e^{-R^2 \cos 2\theta}$$

and we have $\cos(2\theta) \geq 1 - \frac{4\theta}{\pi} \forall \theta \in [0, \pi/4]$

and $|I_{BC}| \leq R e^{R^2} \underbrace{\int_0^{\pi/4} e^{+R^2 \frac{4}{\pi} \theta} d\theta}_{\left[\frac{\pi}{4R^2} e^{R^2 \frac{4}{\pi} \theta} \right]_0^{\pi/4}} = \frac{\pi}{4R} (1 - e^{-R^2}) \xrightarrow[R \rightarrow \infty]{} 0$

$$\left[\frac{\pi}{4R^2} e^{R^2 \frac{4}{\pi} \theta} \right]_0^{\pi/4} = \frac{\pi}{4R^2} (e^{+R^2} - 1)$$

so $|I_{BC}| \rightarrow 0$

$$\bullet I_{CA} = \frac{1+i}{\sqrt{2}} \int_0^R dr e^{-r^2 \frac{1}{2}(1+i)^2} = \frac{1+i}{\sqrt{2}} \int_0^R dr [\cos r^2 - i \sin r^2], \text{ thus}$$

$$\frac{1+i}{\sqrt{2}} \left(\int_0^\infty \cos r^2 dr - i \int_0^\infty \sin r^2 dr \right) = \frac{\sqrt{\pi}}{2} \quad \text{give} \quad \frac{1}{\sqrt{2}}(I+J) + \frac{i}{\sqrt{2}}(I-J) = \frac{\sqrt{\pi}}{2}$$

identifying Real and Imaginary parts gives

$$I = J = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

4. Legendre function:

a. taking $a=t$ and $f(z) = (z^2-1)^n$ in Cauchy, C centered on t \rightarrow entire function

$$P_n(t) = \frac{1}{2^n n!} \frac{n!}{2\pi i} \oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}} dz$$

b. we write $z = t + \sqrt{t^2-1} e^{i\theta}$  , $|t| > 1$

then $z^2-1 = t^2 + 2t\sqrt{t^2-1} e^{i\theta} + (t^2-1)e^{2i\theta} - 1$

$$= 2\sqrt{t^2-1} e^{i\theta} (t + \sqrt{t^2-1} \cos\theta) \text{ and } z-t = \sqrt{t^2-1} e^{i\theta} \quad dz = \sqrt{t^2-1} e^{i\theta} d\theta$$

$$\therefore P_n(t) = \frac{1}{2^n} \frac{1}{2\pi i} \int_0^{2\pi} \frac{(\sqrt{t^2-1} e^{i\theta})^n (t + \sqrt{t^2-1} \cos\theta)^n}{(\sqrt{t^2-1} e^{i\theta})^{n+1}} \cdot \sqrt{t^2-1} e^{i\theta} d\theta$$

$$P_n(t) = \frac{1}{2\pi} \int_0^{2\pi} (t + \sqrt{t^2-1} \cos\theta)^n d\theta$$

Integrals using residues

1. $I = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ with $|a| > |b|$, usual technique for

integrals of the type $I = \int G(\cos\theta, \sin\theta) d\theta$, set $z = e^{i\theta}$ on ℓ $\frac{dz}{iz} = d\theta$ $\Rightarrow |z|=1$

$$\text{here, } I = \frac{1}{2\pi} \oint_{\ell} \frac{dz}{z} \frac{1}{a + \frac{b}{2\pi} (z - \frac{1}{z})}$$

$$= \frac{1}{\pi} \oint \frac{dz}{bz + 2az - b}$$

factorization: $\Delta' = (ia)^2 - b(-b)$

$$= b^2 - a^2 = (\pm i\sqrt{a^2 - b^2})$$

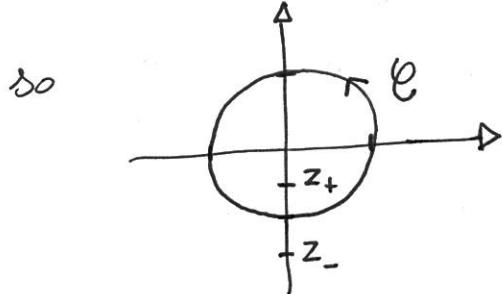
roots are $z_{\pm} = \frac{1}{b} [-ia \pm i\sqrt{a^2 - b^2}] \approx$ pure imaginary

$$|z_{\pm}| = \frac{1}{|b|} |-a \pm \sqrt{a^2 - b^2}|$$

$$\Rightarrow |z_-| = |a + \sqrt{a^2 - b^2}| / |b| > 1$$

along the imaginary axis on the < 0 part.

$$|z_+| = \frac{1}{|b|} \sqrt{|a^2 - b^2|} - a \dots \text{ simpler: } z_+ z_- = -1 \Rightarrow |z_+| = \frac{1}{|z_-|} < 1$$



the pole in the interior of C is z_+

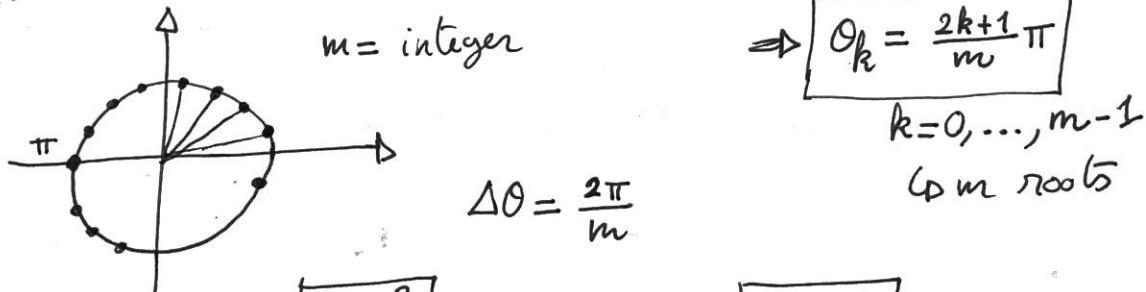
$$\begin{aligned} \operatorname{Res}(f, z_+) &= \lim_{z \rightarrow z_+} (z - z_+) \frac{1}{\pi} \frac{1}{b(z - z_+)(z - z_-)} \\ &= \frac{1}{\pi b} \frac{1}{z_+ - z_-} = \frac{1}{\pi b} \frac{1}{2i \sqrt{a^2 - b^2}} \end{aligned}$$

Applying the residue theorem:

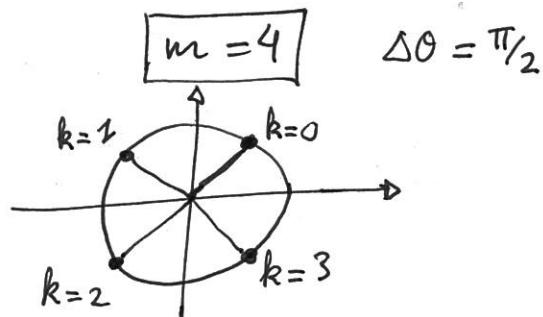
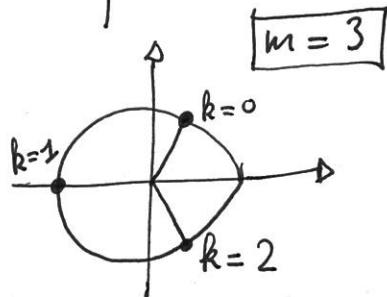
$$I = 2\pi i \times \frac{1}{2\pi i} \frac{1}{\sqrt{a^2 - b^2}} \Rightarrow I = \frac{1}{\sqrt{a^2 - b^2}}$$

2. $I = \int_0^\infty \frac{dx}{1+x^m}$; the poles of $f(z) = \frac{1}{1+z^m}$ are such

that $z_k^m = -1 = e^{i\pi}$ so $z_k = e^{i\theta_k}$ and $m\theta_k = \pi + 2k\pi$



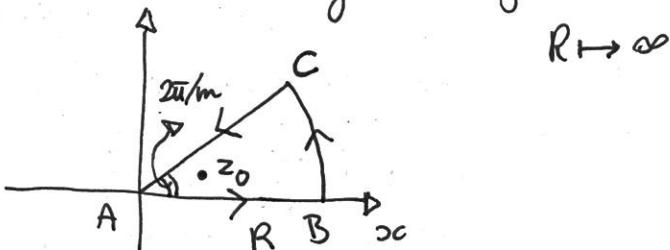
example:



idea: $z=1$ never a pole and always a single pole in $\theta \in [0, \frac{2\pi}{m}]$

which is $z_0 = e^{i\pi/m}$

Choose the following contour enclosing z_0 :



according to the residue theorem:

$$\oint_{\mathcal{C}} \frac{dz}{1+z^m} = 2i\pi \operatorname{Res}(z_0)$$

first we must compute the residue using previous result

$$P(z) = 1, \quad Q(z) = 1+z^m, \quad Q'(z) = m z^{m-1}$$

$$\text{so that } \operatorname{Res}(z_0) = m^{-1} (e^{i\pi/m})^{m-1} = m^{-1} e^{-i\pi} e^{+i\pi/m} = -m^{-1} e^{i\pi/m}$$

The contour integral reads:

- $I_{AB} = \int_0^R \frac{dx}{1+x^m} \rightarrow I \text{ as } R \rightarrow \infty$
- $I_{BC} = iR \int_0^{2\pi/m} d\theta \frac{e^{i\theta}}{1+R^m e^{im\theta}}$

$z = R e^{i\theta}$
 $dz = iR e^{i\theta} d\theta$

we could use Jordan's lemma (if f is holomorphic in $\mathbb{D} \subset \mathbb{C}$ and if $|z f(z)| \rightarrow 0$ as $|z| \rightarrow \infty$, then $\oint_{\text{circle}(r, 0) \cap \mathbb{D}} f(z) dz = 0$ as $r \rightarrow \infty$)

or by hand

$$|I_{BC}| \leq R \int_0^{2\pi/m} d\theta \frac{1}{(1+R^{2m} + 2\cos(m\theta) R^m)^{1/2}} \leq \frac{R}{1+R^m} \xrightarrow{R \rightarrow \infty}$$

$\hookrightarrow \max \text{ for } \theta=0$

$$I_{CA} = e^{i2\pi/m} \int_R^0 dr \frac{1}{1+r^m (e^{i2\pi/m})^m} \text{ as } z = r e^{i2\pi/m} \quad r \in [R, 0]$$

$$= -e^{i2\pi/m} \int_0^R \frac{dr}{1+r^m}$$

$\underbrace{\hspace{1cm}}_1$
 $\underbrace{\hspace{1cm}}_0$
 $= I \text{ as } R \rightarrow \infty$

$$\text{so finally } (1 - e^{i2\pi/m}) I = -2i\pi e^{i\pi/m}$$

$$\text{or } I = \frac{2i\pi/m}{e^{i\pi/m} - e^{-i\pi/m}}$$

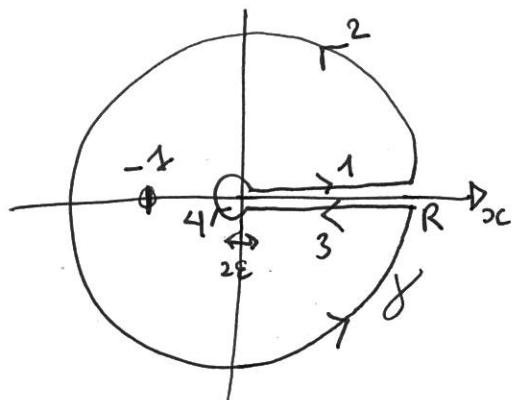
$\Rightarrow I = \frac{\pi/m}{\sin(\pi/m)}$

$$2. (\text{bis}) \quad Q(z) \simeq Q(z_0) + (z-z_0) Q'(z_0) + \dots \simeq (z-z_0) Q'(z_0) + \dots$$

as z_0 is a simple pole, we have $Q(z_0) = 0$
and $Q'(z_0) \neq 0$

$$\text{so } \lim_{z \rightarrow z_0} (z-z_0) \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q'(z_0)}.$$

3. We consider $f(z) = \frac{z^{\delta-1}}{1+z}$, $0 < \delta < 1$ and the contour



$$\text{we write } z^{\delta-1} = e^{(\delta-1)\ln z}$$

we take the following determination of the logarithm: $\arg(z) \in [0, 2\pi[$

so that: on 1 $\Rightarrow \ln z = \ln x$

on 3 $\Rightarrow \ln z = \ln x + 2i\pi$

$f(z)$ is meromorphic in a domain containing γ with pole at $z_0 = -1$

$$\text{so } I_1 + I_2 + I_3 + I_4 = 2i\pi \operatorname{Res}(-1) = 2i\pi (-1)^{\delta-1} \\ = 2i\pi e^{(\delta-1)\ln(-1)} = 2i\pi e^{i(\delta-1)} = -2i\pi e^{-i\pi}$$

with $\ln(-1) = \ln|-1| + i\pi = i\pi$ with the chosen determination

$$\bullet I_1 = \int_{-\infty}^R \frac{z^{\delta-1} dx}{1+x} \xrightarrow[\epsilon \rightarrow 0]{} I \text{ as } R \rightarrow \infty, \epsilon \rightarrow 0$$

$$\bullet |I_2| \leq \int_0^{2\pi} d\theta \frac{|z|^{\delta-1}}{|1+z|} \text{ with } z = Re^{i\theta} \text{ for } R \rightarrow \infty$$

$$\bullet I_3 = \int_R^\epsilon \frac{e^{(\delta-1)\ln z}}{1+z} dz \text{ with } z = x e^{i2\pi} \\ \ln z = \ln x + 2i\pi \sim 2\pi R^{\delta-2} \rightarrow 0 \text{ since } \delta < 1$$

$$= - \int_c^R dx \frac{e^{(\delta-1)(\ln x + 2i\pi)}}{1+2x}$$

$$I_3 = - e^{2i\pi(s-1)} I \text{ when } R \rightarrow \infty, \epsilon \rightarrow 0$$

$$= - e^{2i\pi s} I$$

$$\cdot I_4 = \int_{2\pi}^0 d\theta i\epsilon \frac{\epsilon^{s-1} e^{i\theta(s-1)} e^{i\theta}}{1 + \epsilon e^{i\theta}} dz = \epsilon e^{i\theta} dz = \epsilon i e^{i\theta} d\theta$$

$$|I_4| \leq 2\pi \epsilon^{s-1+1} \leq 2\pi \epsilon^s \rightarrow 0 \text{ as } s > 0$$

finally: $I(1 - e^{2i\pi s}) = -2i\pi e^{i\pi s}$

or $I = \frac{2i}{e^{i\pi s} - e^{-i\pi s}} \pi \rightarrow \boxed{I = \frac{\pi}{\sin(\pi s)}}$

Rk: this generalizes the previous result:

$$\int_0^\infty dt \frac{t^{s-1}}{1+t} = \frac{1}{s} \int_0^\infty \frac{dx}{x} \frac{(x^{1/s})^s}{1+x^{1/s}} = \frac{1}{s} \int_0^\infty dx \frac{1}{1+x^{1/s}}$$

$$\begin{aligned} t &= x^{1/s} \\ \frac{dt}{t} &= \frac{1}{s} \frac{dx}{x} \end{aligned}$$

by setting $\alpha = 1/s > 1$, we have

$$\boxed{\int_0^\infty dx \frac{1}{1+x^\alpha} = \frac{\pi/\alpha}{\sin(\pi/\alpha)}}$$

Integrals with Γ and B functions

Preliminaries:

Let us show that

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

we start from:

$$\Gamma(p)\Gamma(q) = 4 \iint_0^\infty x^{2p-1} y^{2q-1} e^{-(x^2+y^2)} dx dy$$

and go to polar coordinates: $x = r \cos \theta \quad \theta \in [0, \pi/2]$
 $y = r \sin \theta$
 $dx dy = r dr d\theta$

$$= 4 \int_0^\infty r^{2(p+q)-2+1-\frac{1}{2}} dr \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta d\theta$$

$\underbrace{\Gamma(p+q)/2}$

and the Wallis integral by setting $\sin \theta = \sqrt{t}$; $\cos \theta = \sqrt{1-t}$; $\cos \theta d\theta = \frac{dt}{2\sqrt{t(1-t)}}$

$$I_{p,q} = \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta d\theta = \int_0^1 (1-t)^{\frac{2p-1}{2}} t^{\frac{2q-1}{2}} \frac{dt}{2\sqrt{t(1-t)}} \quad d\theta = \frac{dt}{2\sqrt{t(1-t)}}$$

$$= \frac{1}{2} \int_0^1 (1-t)^{p-1} t^{q-1} dt = \frac{1}{2} B(p, q)$$

finally, we obtain the formula!

We recover in passing the result of the first exercise.

① Complement formula:

same strategy:

$$\boxed{\Gamma(x)\Gamma(1-x) = \underbrace{\Gamma(1)}_{\substack{\\ \Downarrow}}}, \quad B(x, 1-x) = \int_0^1 (1-t)^{x-1} t^{1-x-1} dt$$

a smart move is $u = \frac{1}{t} - 1 \in [0, \infty[\Rightarrow du = -\frac{dt}{t^2} \Rightarrow dt = -\frac{du}{(1+u)}$

$$= \frac{1-t}{t}$$

$$t = \frac{1}{1+u}; 1-t = ut = \frac{u}{1+u}$$

$$= \int_0^\infty du (1+u)^x \left(\frac{u}{1+u}\right)^{x-1} \frac{1}{(1+u)^2} = \int_0^\infty du \frac{u^{x-1}}{(1+u)^{2+x-1-x}}$$

$$= \int_0^\infty du \frac{u^{x-1}}{1+u} = \boxed{\frac{\pi}{\sin(\pi x)}}$$

$$\textcircled{2} \quad \bullet \text{ set } t = x^n, \quad n x^{n-1} dx = dt \quad \textcircled{2} \quad \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \int_0^1 \frac{dt}{n t^{(n-1)/n}} \frac{1}{(1-t)^{1/n}}$$

• requires $|x| < 1$

$$\text{set } t = \frac{1-x}{1+x}; dt = \frac{-(1+x)-(1-x)}{(1+x)^2} dx$$

$$(1+x)t = 1-x$$

$$= -2 \frac{dx}{(1+x)^2}$$

$$x = \frac{1-t}{1+t} \quad = -2 \left(\frac{1+t+1-t}{1+t} \right)^2 dx$$

$$t \in [0, \infty[\quad = -\frac{1}{2} (1+t)^2 dx$$

$$= \int_0^1 dt t^{\frac{1}{n}-1} (1-t)^{\frac{1}{n}}$$

$$= \frac{1}{n} B\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$$

$$= \frac{1}{n} \Gamma\left(\frac{1}{n}\right) \Gamma\left(1 - \frac{1}{n}\right)$$

$$= \frac{\pi/n}{\sin(\pi/n)}$$

$$\textcircled{2} \quad \boxed{\int_{-1}^1 \left(\frac{1-x}{1+x}\right)^\alpha dx = \int_0^\infty dt \frac{2}{(1+t)^2} t^\alpha \stackrel{\text{IPP}}{=} 2 \left\{ \left[-\frac{1}{1+t} t^\alpha \right]_0^\infty + \int_0^\infty dt \frac{2t^{\alpha-1}}{1+t} \right\} = \frac{2\alpha\pi}{\sin(\pi\alpha)}}$$