

A useful relation with Gaussian distribution

1. we write

$$\begin{aligned} \frac{\partial}{\partial x_j} \left(F(\vec{x}) P(\vec{x}) \right) &= \frac{\partial F}{\partial x_j} P(\vec{x}) + F(\vec{x}) \frac{\partial P}{\partial x_j} \\ &= \frac{\partial F}{\partial x_j} P(\vec{x}) - F(\vec{x}) \left(\sum_k A_{jk} x_k \right) P(\vec{x}) \end{aligned}$$

$$\begin{aligned} \text{so } \left\langle \frac{\partial F}{\partial x_j} \right\rangle &= \int d\vec{x} \frac{\partial F}{\partial x_j} P(\vec{x}) = \underbrace{\int d\vec{x} \frac{\partial}{\partial x_j} (F(\vec{x}) P(\vec{x}))}_{+ \infty} + \sum_k A_{jk} \underbrace{\int d\vec{x} x_k F(\vec{x}) P(\vec{x})}_{\langle x_k F(\vec{x}) \rangle} \\ &= \int_{-\infty}^{+\infty} dx_j \frac{\partial}{\partial x_j} G(x_j) = 0 \end{aligned}$$

where $G(x_j) = \prod_{i \neq j} \int dx_i F(\vec{x}) P(\vec{x})$ and since $\forall \vec{x}, \vec{x}^T A \vec{x} > 0$
 $\Rightarrow x_j A_{jj} x_j > 0$

in the exponential, this term kills $G(x_j)$ ↪
 when $x_j \rightarrow \pm \infty$

Finally:

$$\left\langle \frac{\partial F}{\partial x_j} \right\rangle = \sum_k A_{jk} \langle x_k F(\vec{x}) \rangle \text{ of the form } b_j = (\vec{A} \vec{c})_j \text{ or } \vec{b} = \vec{A} \vec{c} \Leftrightarrow \vec{c} = \vec{A}^{-1} \vec{b}$$

$$\text{so } \langle x_k F(\vec{x}) \rangle = \sum_j \underbrace{[\vec{A}^{-1}]_{kj}}_{\langle x_k x_j \rangle} \langle \frac{\partial F}{\partial x_j} \rangle \quad \text{Q.E.D.}$$

2. we want to compute (n even):

$$\begin{aligned} \langle x_{k_1} \cdots x_{k_n} \rangle &= \langle x_{k_n} F(\vec{x}) \rangle = \sum_j \underbrace{\langle x_{k_n} x_j \rangle}_{\text{F}(\vec{x}) = x_{k_1} \cdots x_{k_{n-1}}} \underbrace{\langle \frac{\partial F}{\partial x_j} \rangle}_{\text{n-2 terms}} \\ &\quad \sum_{k_d} \delta_{ik_d} \prod_{\substack{k \neq k_d \\ k \in \{k_1, \dots, k_{n-1}\}}} x_k \end{aligned}$$

$$\begin{aligned} &= \sum_{\substack{k_d \\ \in \{k_1, \dots, k_{n-1}\}}} \underbrace{\langle x_{k_n} x_{k_d} \times \prod_{\substack{k \neq k_d \\ k \in \{k_1, \dots, k_{n-1}\}}} x_k \rangle}_{\text{can be treated by Wick's theorem}} \\ &\quad \text{Diagram: two boxes connected by a line, each with two dots.} \end{aligned}$$

Thus, the proof goes by induction.

Practical uses of Wick's theorem:

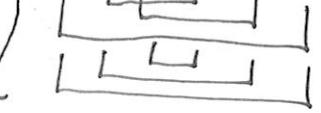
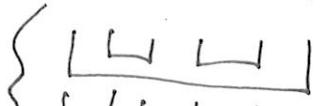
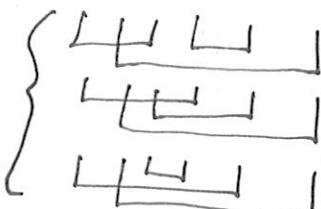
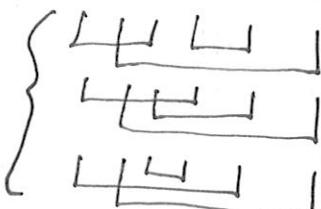
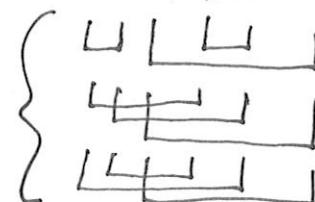
1. $n=2 \quad \langle x \bar{c}_{h_1} x \bar{c}_{h_2} \rangle$ 1 term

$n=4 \quad \langle x \bar{c}_{h_1} x \bar{c}_{h_2} x \bar{c}_{h_3} x \bar{c}_{h_4} \rangle$ 3 terms

$n=6 \quad \langle x \bar{c}_{h_1} x \bar{c}_{h_2} x \bar{c}_{h_3} x \bar{c}_{h_4} x \bar{c}_{h_5} x \bar{c}_{h_6} \rangle$

3 previous terms {

3 more { Total of 15 terms



recursively, let t_n be the number of terms in the expansion

$$t_2 = 1, \quad t_4 = 3, \quad t_6 = 15 = 5 \times t_4; \text{ number } \underbrace{\langle x \dots x \rangle}_{n \text{ factors}} = \sum_{j=1}^{n-1} \langle \cancel{x_j} \cancel{x_{j+1}} \dots \cancel{x_n} \rangle \underbrace{\langle x \dots x \rangle}_{n-2 \text{ factors}}$$

$$= 5 \cdot 3 \cdot 1$$

on has $t_n = (n-1) \times t_{n-2}$

$$\boxed{t_n = (n-1)(n-3)\dots 3 \cdot 1} = \frac{n!}{n(n-2)\dots 2} = \frac{(2p)!}{2^p p!} \quad \text{if } n=2p$$

2. from Gamma's function, we had

$$I_n = \int_0^\infty dx x^n e^{-\frac{1}{2}ax^2} = \frac{1}{2} \left(\frac{2}{a}\right)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right);$$

$$\text{if } p(x) = \sqrt{\frac{a}{2\pi}} e^{-\frac{1}{2}ax^2} \quad \text{with for } n=0; \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \quad I_0 = \frac{1}{2} \sqrt{\frac{2\pi}{a}}$$

$$\begin{aligned} \langle x^n \rangle &= \int_{-\infty}^{+\infty} dx x^n p(x) = \left(\frac{a}{2}\right)^{n/2} \frac{1}{\Gamma\left(\frac{1}{2}\right)} 2 I_n \\ &= \left(\frac{2}{a}\right)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \end{aligned}$$

$$\begin{aligned} \text{and } \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) &= \left(\frac{n}{2} - \frac{1}{2}\right) \Gamma\left(\frac{n}{2} - \frac{1}{2}\right) = \frac{n-1}{2} \cdot \frac{n-3}{2} \cdots \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\ &= \frac{t_n}{2^{n/2}} \Gamma\left(\frac{1}{2}\right) \end{aligned}$$

$$\text{so } \boxed{\langle x^n \rangle = \frac{2^{n/2}}{a^{n/2}} \frac{t_n}{2^{n/2}} \frac{\Gamma(1/2)}{\Gamma(n/2)} = t_n (1/a)^{n/2} = \boxed{t_n \langle x^2 \rangle^{n/2}}} \quad \begin{array}{l} \text{Variance:} \\ a = 1/\sigma^2 \\ \langle x^2 \rangle = \sigma^2 \end{array}$$

• from Wick's theorem:

$$\boxed{\langle x^n \rangle = \langle x \cdot x \cdots x \rangle = \sum_{P \in S(n)} \underbrace{\langle x \cdot x \rangle \cdots \langle x \cdot x \rangle}_{\substack{n/2 \text{ terms in the product} \\ t_n \text{ terms}}} = \boxed{t_n \langle x^2 \rangle^{n/2}}} \quad \begin{array}{l} \text{all identical.} \end{array}$$

much simpler to remember \Rightarrow practical for small n : $\langle x^4 \rangle = 3 \langle x^2 \rangle^2$
 $\langle x^6 \rangle = 15 \langle x^2 \rangle^3$

3. a) the correlation matrix:

$$\Delta = \langle x_i x_j \rangle = \begin{pmatrix} a & b & c \\ b & a & c \\ c & c & a \end{pmatrix} = A^{-1}$$

$$\cdot \langle x^3 y \rangle = \langle \underbrace{x \cdot x \cdot x}_4 \underbrace{y}_1 \rangle = \langle x^2 \rangle \langle xy \rangle + \langle x^2 \rangle \langle xy \rangle + \langle x^2 \rangle \langle xy \rangle = \underline{3ab}$$

$$\cdot \langle x^2 y^2 \rangle = \langle x^2 \rangle \langle y^2 \rangle + 2 \langle xy \rangle \langle xy \rangle = \underline{a^2 + 2b^2}$$

$$\cdot \langle x^2 y z \rangle = \langle x^2 \rangle \langle yz \rangle + 2 \langle xy \rangle \langle xz \rangle = \underline{ac + 2bc}$$

b) in the case $c=0$, we can easily invert the Δ matrix to get A :

$$\Delta = \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix} = A^{-1}; \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2-b^2} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}; \quad |b| \neq a > 0$$

[Recall that: $\det M \neq 0$,
 if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow M^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$]

we get

$$A = \frac{1}{a^2-b^2} \begin{pmatrix} a & -b & 0 \\ -b & a & 0 \\ 0 & 0 & \frac{a^2-b^2}{a} \end{pmatrix}$$

so $p(x, y, z) = \exp \left\{ \frac{-1}{a^2-b^2} \left(\frac{a(x^2+y^2)}{2} + \frac{a^2-b^2}{2a} z^2 - bxy \right) \right\} \times \sqrt{\frac{\det A}{(2\pi)^3}}$

with $\det A = \frac{1}{\det(A^{-1})} = a(a^2-b^2) \left(= a \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} - b \begin{vmatrix} b & 0 \\ 0 & a \end{vmatrix} \right)$