

Properties of Pauli matrices

1. The following properties are deduced by looking at the matrices:

$$\sigma_i^\dagger = \sigma_i, \quad \sigma_i^2 = 1, \quad \text{Tr}(\sigma_i) = 0, \quad \det(\sigma_i) = -1$$

by an explicit calculation, one gets:

$$\sigma_1 \sigma_2 = i \sigma_3, \quad \sigma_2 \sigma_1 = -i \sigma_3, \quad \sigma_1 \sigma_3 = -i \sigma_2, \quad \sigma_3 \sigma_1 = i \sigma_2, \quad \sigma_2 \sigma_3 = i \sigma_1, \quad \sigma_3 \sigma_2 = -i \sigma_1$$

from which one gets $[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$ and $\begin{cases} \sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k \text{ for } i \neq j \\ \sigma_i^2 = 1 \end{cases}$

$$\{\sigma_i, \sigma_j\} = 2 \delta_{ij}$$

2. combining the two gives: $[\sigma_i, \sigma_j] + \{\sigma_i, \sigma_j\} = 2 \sigma_i \sigma_j = 2 \delta_{ij} + i \epsilon_{ijk} \sigma_k$

or $\boxed{\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k}$ linearization of products

and $\boxed{(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \sum_{ij} a_i b_j \sigma_i \sigma_j = \left(\sum_{ij} a_i b_j \delta_{ij} \right) \mathbb{1} + i \sum_{ij} a_i b_j \epsilon_{ijk} \sigma_k = (\vec{a} \cdot \vec{b}) \mathbb{1} + i (\vec{a} \wedge \vec{b}) \cdot \vec{\sigma}}$

$$\sum_i a_i b_i = \sum_{ij} \epsilon_{kij} a_i b_j = (\vec{a} \wedge \vec{b})_k$$

R_k : from $\begin{cases} \sigma_k^2 = 1 \\ \sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k \end{cases}$, we get $\sigma_i \sigma_j \sigma_k = i \mathbb{1}$ for $i \neq j \neq k$

3. $U_{\vec{n}}(\psi) = e^{-i\psi \vec{n} \cdot \vec{\sigma}} = \sum_{k=0}^{\infty} \frac{(-i\psi)^k}{k!} (\vec{n} \cdot \vec{\sigma})^k$ with $(\vec{n} \cdot \vec{\sigma})^2 = \frac{1}{4} (\vec{n} \cdot \vec{\sigma})^2 = \frac{1}{4} \vec{n} \cdot \vec{n} \mathbb{1} + i (\vec{n} \wedge \vec{n}) \cdot \vec{\sigma}$

$$= \sum_{p=0}^{\infty} \frac{(-i)^p (\psi)^{2p}}{(2p)!} \left(\frac{1}{4} \right)^p \mathbb{1} + (-i) \sum_{p=0}^{\infty} \frac{(-i)^p (\psi/2)^{2p+1}}{(2p+1)!} \vec{n} \cdot \vec{\sigma} = \frac{1}{4} \mathbb{1} \Rightarrow (\vec{n} \cdot \vec{\sigma})^{2p} = \left(\frac{1}{4} \right)^p = \frac{1}{2^{2p}}$$

$$(\vec{n} \cdot \vec{\sigma})^{2p+1} = \frac{1}{2^{2p+1}} \vec{n} \cdot \vec{\sigma}$$

$$\boxed{U_{\vec{n}}(\psi) = \cos\left(\frac{\psi}{2}\right) \mathbb{1} - i \sin\left(\frac{\psi}{2}\right) \vec{n} \cdot \vec{\sigma}} \quad \Delta \quad \psi/2 \text{ and not } \psi!$$

4. let us check that $(M, N) = \frac{1}{2} \text{Tr}(M^\dagger N)$ is a scalar product on the space of complex matrices.

* $(M, N) = (N, M)^*$ hermitic

* $\begin{cases} (\lambda M + \mu N, O) = \lambda^* (M, O) + \mu^* (N, O) \\ (M, \lambda N + \mu O) = \lambda (M, N) + \mu (M, O) \end{cases}$ sesquilinear

* positive: $(M, M) = \|M\|^2 = \frac{1}{2} \text{Tr}(M^\dagger M) = \frac{1}{2} \sum_{ij} (M^\dagger)_{ij} M_{ji} = \frac{1}{2} \sum_{ij} M_{ji}^* M_{ji} = \frac{1}{2} \sum_{ij} |M_{ij}|^2 \geq 0$

* definite: if $\|M\| = 0$, then $M = 0$ (from above line)

Decomposition of a matrix over the Pauli matrices.

The dimension of $GL(2)$ on complex numbers is 4 and we have

$$\text{if } i, j \neq 0 \quad \boxed{(\sigma_i, \sigma_j) = \frac{1}{2} \text{Tr}(\sigma_i^\dagger \sigma_j) = \frac{1}{2} \text{Tr}(\sigma_i \sigma_j) = \frac{1}{2} \text{Tr}(\mathbb{1}) \delta_{ij} + i \varepsilon_{ijk} \text{Tr}(\sigma_k) = \delta_{ij}}$$

$$\text{and } (\sigma_i, \sigma_0) = \frac{1}{2} \text{Tr}(\sigma_i) = 0 \text{ since } \sigma_0 = \mathbb{1}$$

$$(\sigma_0, \sigma_0) = \frac{1}{2} \text{Tr}(\sigma_0^2) = 1$$

So $\{\sigma_i\}_{i=0, \dots, 3}$ forms an orthonormal basis of 2×2 complex matrices

$$\text{For any } H: \quad \boxed{H = \sum_{i=0}^3 (\sigma_i, H) \sigma_i = \sum_{i=0}^3 \frac{1}{2} \text{Tr}(\sigma_i H) \sigma_i}$$

5. a) using $C^2 = \mathbb{1} \Rightarrow C^{-1} = C$; $\sigma_i^T = \begin{cases} \sigma_i & \text{for } i=1,3 \\ -\sigma_2 & \text{for } i=2 \end{cases}$, we have

$$\boxed{C^{-1} \sigma_i^T C = -C \sigma_i^T C = +\sigma_2 \sigma_i^T \sigma_2 = \begin{cases} \sigma_2 \sigma_i \sigma_2 = -\sigma_i & \text{for } i=1,3 \\ -\sigma_2^3 = -\sigma_2 & \text{for } i=2 \end{cases}} \\ = -\sigma_i \quad \text{or} \quad \boxed{C^{-1} \sigma_i^* C = -\sigma_i}$$

b) $U \in SU(2) \Rightarrow U^\dagger U = U U^\dagger = \mathbb{1}$ and $\det(U) = 1$

we consider explicitly

$$C^{-1} U^T C U = - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = - \begin{pmatrix} u_{12} & u_{22} \\ u_{11} & -u_{21} \end{pmatrix} \begin{pmatrix} u_{21} & u_{22} \\ -u_{11} & u_{12} \end{pmatrix} = \det(U) \mathbb{1} = \mathbb{1}$$

$$\text{so } \boxed{C^{-1} U^T C = U^{-1} = U^\dagger}$$

c) From the decomposition of U over $\{\sigma_i\}_{i=0, \dots, 3}$, we have

$$U = u_0 - i \vec{u} \cdot \vec{\sigma} \quad \text{with } u_i \in \mathbb{C}$$

we have to show that the $u_i \in \mathbb{R}$. We write

$$C^{-1} (u_0 - i \vec{u} \cdot \vec{\sigma})^T C = u_0^* + i \vec{u}^* \cdot \vec{\sigma} = u_0 + i \vec{u} \cdot \vec{\sigma}$$

by taking the scalar product (σ_i, \cdot) on this equality one gets $u_i^* = u_i$

$$\text{Last } U = \begin{pmatrix} u_0 - i u_3 & -u_2 - i u_1 \\ u_2 - i u_1 & u_0 + i u_3 \end{pmatrix} \text{ so } \det(U) = u_0^2 + u_3^2 - (-u_2^2 - u_1^2) = \boxed{u_0^2 + \vec{u}^2 = 1}$$

d) As we have seen in the lecture, one can choose $\begin{cases} u_0 = \cos \frac{\Psi}{2} & \Psi \in [0, 2\pi] \\ \vec{u} = \vec{n} \sin \frac{\Psi}{2} & \vec{n}^2 = 1 \end{cases}$