

Dirac distribution

$$1. \quad \delta(x^2 - a^2) = \frac{1}{|2a|} \sum_{x_i=\pm a} \delta(x - x_i) = \frac{1}{|2a|} \left\{ \delta(x-a) + \delta(x+a) \right\}$$

$\uparrow \quad f'(x_i) = 2x_i$

distribution sense:

$$\int dx f(x) \delta(x^2 - a^2) = \frac{1}{|2a|} \left\{ f(a) + f(-a) \right\} \underset{\substack{a \rightarrow 0 \\ f \text{ continuous}}}{\approx} \frac{f(0)}{|a|} = \int dx f(x) \underbrace{\frac{\delta(x-a)}{|x|}}_{\approx \frac{\delta(x)}{|x|}}$$

this can be understood in the sense of distribution as

$$\boxed{\delta(x^2) = \frac{\delta(x)}{|x|}}$$

understood has no divergence of the integral because of $x=0$

2. zeroes of $\sin(x)$ are $x_n = n\pi$, $n \in \mathbb{Z}$, since $|\sin'(x_n)| = |\cos(x_n)| = 1$

we have $\boxed{\delta(\sin(x)) = \sum_{n=-\infty}^{\infty} \delta(x-n\pi)}$

$$3. a) \quad \rho(\epsilon) = \int_{\mathbb{R}^d} d\vec{k} \delta(E - (\Delta + c \|\vec{k}\|^p)) = \int_0^\infty dk S_d k^{d-1} \delta(E - (\Delta + c k^p))$$

↑ spherical

• if $E < \Delta$: the equation

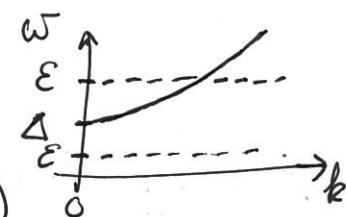
$f(k) = E - \Delta - ck^p = 0$ has no solution (no states!)

$$\Rightarrow \boxed{\rho(\epsilon) = 0}$$

• if $E \geq \Delta$: there is one solution with positive k : $k_+ = \left(\frac{E-\Delta}{c}\right)^{1/p}$

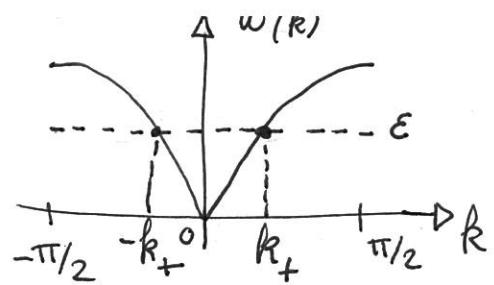
then using: $\delta(f(k)) = \frac{1}{|f'(k)|} \delta(k - k_+)$ with $f'(k) = -cpk^{p-1}$ we have

$$\boxed{\rho(\epsilon) = \frac{S_d}{cp} \int_0^\infty dk \frac{k^{d-1}}{k^{p-1}} \delta(k - k_+) = \frac{S_d}{cp} k_+^{d-p} = \frac{S_d}{cp} \left(\frac{E-\Delta}{c}\right)^{\frac{d}{p}-1}}$$



b) we have

$$\rho(\epsilon) = \int_{-\pi/2}^{\pi/2} dk \delta(\epsilon - c \sin k)$$

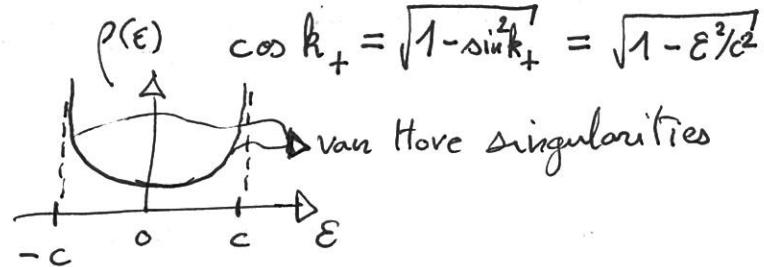


$$= \frac{2}{c} \int_0^{\pi/2} dk \delta(k - k_+) \frac{1}{|\cos k_+|}$$

$$f(k) = \epsilon - c \sin k \Rightarrow \sin k_+ = \frac{\epsilon}{c}$$

$$f'(k) = -c \cos k$$

$$\boxed{\rho(\epsilon) = \frac{2}{\sqrt{c^2 - \epsilon^2}}}$$

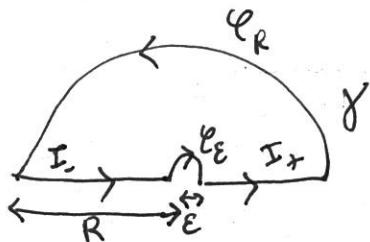


Low-energy: $\rho(\epsilon) \approx \frac{2}{c}$ is constant

in a); at low ϵ : $\omega(k) \approx ck$ (relativistic) and $d=p=1, \Delta=0, S_i=2$
 $\Rightarrow \rho(\epsilon) = \frac{2}{c} \approx \underline{\underline{OK!}}$

Integral representations of the Heaviside function

Let $f(z) = e^{izt}$, we consider first $t > 0$: $\frac{e^{izt}}{z}$ being holomorphic within γ



$$\oint_{\gamma} \frac{e^{izt}}{z} dz = 0 = \int_{-\infty}^{+\infty} dw \frac{e^{iwt}}{w} + \int_{C_E} \frac{f(z)}{z} dz + \int_{C_R} \frac{f(z)}{z} dz$$

$$\text{pp} \int_{-\infty}^{+\infty} dw \frac{e^{iwt}}{w}$$

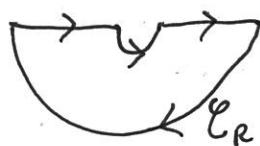
$$R \rightarrow \infty$$

Sorin's lemma

$$\int_0^\infty d\theta \frac{iEe^{i\theta}}{\pi} x e^{ix} \xrightarrow{x \rightarrow 0} -i\pi f(0)$$

$$\text{we get } \text{pp} \int_{-\infty}^{+\infty} dw \frac{e^{iwt}}{w} = i\pi \text{ if } t > 0$$

if $t < 0$: we take the contour (to kill C_R)



$$0 = \text{pp} \int_{-\infty}^{+\infty} dw \frac{e^{iwt}}{w} + 0 + i\pi f(0)$$

last if $t=0$: $f(z)=1$; the contribution of ϵ_R does not vanish

$$\oint \frac{dz}{z} = 0 = \text{pp} \int_{-\infty}^{+\infty} \frac{dw}{w} - i\pi \cdot 1 + i \int_0^\pi d\theta$$

$\epsilon_R = \pi$

which gives $\text{pp} \int_{-\infty}^{+\infty} \frac{dw}{w} = 0$; otherwise $\int_{\epsilon}^{+\infty} \frac{dw}{w} + \int_{-\infty}^{-\epsilon} \frac{dw}{w} = 0 \forall \epsilon$.

finally:

$$\frac{1}{2i\pi} \text{pp} \int_{-\infty}^{+\infty} dw \frac{e^{iwt}}{w} = \begin{cases} 1/2 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1/2 & \text{if } t < 0 \end{cases}$$

$$= \frac{1}{2} (2 \Theta(t) - 1)$$

so $\boxed{\Theta(t) = \frac{1}{2} + \frac{1}{2i\pi} \text{pp} \int_{-\infty}^{+\infty} dw \frac{e^{iwt}}{w}}$

Application: take $t=1$: $1 = \frac{1}{2} + \frac{1}{2i\pi} \text{pp} \int_{-\infty}^{+\infty} dw \frac{e^{i\omega}}{\omega}$

as $\text{pp} \int_{-\infty}^{+\infty} \frac{\cos \omega}{\omega} dw = 0$ by symmetry (function is odd)

and $\text{pp} \int_{-\infty}^{+\infty} \frac{\sin \omega}{\omega} dw = \int_{-\infty}^{+\infty} \frac{\sin \omega}{\omega} dw$ (no problem of divergence at $\omega=0$! well-defined)

one gets

$$\frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin \omega}{\omega} dw \quad \text{or}$$

$$\boxed{\int_{-\infty}^{+\infty} \frac{\sin \omega}{\omega} dw = \pi}$$

With Feynman's notation:

from the lecture: $\frac{1}{\omega - i\epsilon} = \text{pp} \frac{1}{\omega} + i\pi \delta(\omega)$

applied to $f(\omega) = e^{i\omega t}$, we get

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} dw \frac{e^{i\omega t}}{\omega - i\epsilon} = \underbrace{\frac{1}{2\pi i} \text{pp} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega} dw}_{\approx \Theta(t) - 1/2} + \underbrace{\frac{i\pi}{2\pi i} \int_{-\infty}^{+\infty} dw \delta(\omega) e^{i\omega t}}_{1/2} = \Theta(t)$$

Kramers - Krönig relations

$f(t)$ is a real function, $X(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t) dt$

$$\text{then, } X^*(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} f^*(t) dt = X(-\omega)$$

$\text{"} f(t) \text{"}$

so if $X(\omega) = X_1(\omega) + iX_2(\omega)$, we have $X_1(\omega) - iX_2(\omega) = X_1(-\omega) + iX_2(-\omega)$

or $\begin{cases} X_1(\omega) = X_1(-\omega) & \text{is even} \\ X_2(\omega) = -X_2(-\omega) & \text{is odd} \end{cases}$

experimentally, $f(t)$ is usually real and $\omega \in [0, \infty[$ is the excitation frequency.

Injecting this in the formula of the lecture gives:

$$\boxed{X_1(\omega) = \frac{1}{\pi} \left(\text{pp} \int_0^\infty \frac{X_2(\omega')}{\omega' - \omega} d\omega' + \text{pp} \int_{-\infty}^0 \frac{X_2(\omega')}{\omega' - \omega} d\omega' \right)}$$

$$\quad \quad \quad \text{pp} \int_0^\infty \frac{-X_2(\omega')}{\omega' + \omega} d\omega'$$

$$= \frac{1}{\pi} \text{pp} \int_0^\infty X_2(\omega') \left\{ \underbrace{\frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega}}_{= 2\omega'/(\omega'^2 - \omega^2)} \right\} d\omega' = \frac{2}{\pi} \text{pp} \int_0^\infty X_2(\omega') \frac{\omega'}{\omega'^2 - \omega^2} d\omega'$$

Similarly

$$\boxed{X_2(\omega) = -\frac{1}{\pi} \left(\text{pp} \int_0^\infty \frac{X_1(\omega')}{\omega' - \omega} d\omega' + \text{pp} \int_0^\infty \frac{X_1(\omega')}{-(\omega' + \omega)} d\omega' \right) \text{ with } \frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega} = \frac{2\omega}{\omega'^2 - \omega^2}}$$

$$= -\frac{2}{\pi} \text{pp} \int_0^\infty d\omega' X_1(\omega') \frac{\omega}{\omega'^2 - \omega^2}$$