

Exercises:

Derivatives 1. using definitions:

$$\bullet F[y] = \int d\vec{x} f(y(\vec{x})) , F[y + \delta y] = \int d\vec{x} f(y(\vec{x}) + \delta y(\vec{x}))$$

$$= \int d\vec{x} \{ f(y(\vec{x})) + f'(y(\vec{x})) \delta y(\vec{x}) \}$$

$$= F[y] + \underbrace{\int d\vec{x} f'(y(\vec{x})) \delta y(\vec{x})}_{\text{! identification}}$$

$$\frac{\delta F}{\delta y(\vec{x})}$$

$$\bullet F[y](\vec{x}') = \int d\vec{x} K(\vec{x}, \vec{x}') y(\vec{x}) \rightarrow \delta F = \int d\vec{x} K(\vec{x}, \vec{x}') \{ y(\vec{x}) + \delta y(\vec{x}) - y(\vec{x}) \}$$

$$= \int d\vec{x} \underbrace{K(\vec{x}, \vec{x}')}_{\frac{\delta F(\vec{x}')}{\delta y(\vec{x})}} \delta y(\vec{x})$$

two-variable
function

$$\bullet F[y](\vec{x}') = f(y(\vec{x}')) \rightarrow \delta F = \int d\vec{x} f(y(\vec{x})) \delta(\vec{x} - \vec{x}')$$

↳ Dirac function!
collision of notation

used
the two
previous
cases

$$\downarrow$$

$$= \int d\vec{x} \underbrace{f'(y(\vec{x})) \delta y(\vec{x})}_{\frac{\delta F(\vec{x}')}{\delta y(\vec{x})}} \delta(\vec{x} - \vec{x}')$$

$$\Rightarrow \frac{\delta F(\vec{x}')}{\delta y(\vec{x})} = f'(y(\vec{x})) \delta(\vec{x} - \vec{x}') \frac{\delta F(\vec{x}')}{\delta y(\vec{x})}$$

$$= f'(y(\vec{x}))$$

$$\bullet F[y] = \int_{\Omega} d\vec{x} (\vec{\nabla} y(\vec{x}))^2 \rightarrow \delta F = \int d\vec{x} \{ \vec{\nabla} \delta y(\vec{x}) \cdot \vec{\nabla} y(\vec{x}) + \vec{\nabla} y(\vec{x}) \cdot \vec{\nabla} \delta y(\vec{x}) \}$$

$$= 2 \int_{\Omega} d\vec{x} \underbrace{\vec{\nabla} y(\vec{x}) \cdot \vec{\nabla} \delta y(\vec{x})}_{\text{not the good form...}}$$

let's integrate by parts (Rk: $\operatorname{div}(\vec{a} \vec{\text{grad}} b) = \vec{\nabla} a \cdot \vec{\nabla} b + a \Delta b$)

$$\vec{\nabla} \vec{y}(\vec{x}) \cdot \vec{\nabla} \delta y(\vec{x}) = \vec{\nabla} \cdot (\delta y(\vec{x}) \vec{\nabla} \vec{y}(\vec{x})) - \delta y(\vec{x}) \vec{\nabla}^2 \vec{y}(\vec{x})$$

so that $\delta F = 2 \int_{\Omega} d\vec{x} \vec{\nabla} \cdot (\delta y(\vec{x}) \vec{\nabla} \vec{y}(\vec{x})) - 2 \int_{\Omega} d\vec{x} \Delta y(\vec{x})$

Gauss-Ostrogradski

$$= \int_{\partial\Omega} d\vec{s} \vec{n}(\vec{s}) \cdot (\delta y(\vec{s}) \vec{\nabla} \vec{y}(\vec{s}))$$



and since $\delta y(\vec{s}) = 0$ from boundary conditions
we obtain

$$\boxed{\frac{\delta F}{\delta y(\vec{x})} = -2 \Delta y(\vec{x})}$$

Using Euler Lagrange equations:

- $f(y(\vec{x})) = f(y, \vec{x}) \Rightarrow \frac{\delta F}{\delta y(\vec{x})} = \frac{\partial f}{\partial y} = f'(y(\vec{x}))$

or $\frac{\delta F}{\delta y(\vec{x})} = f'(y(\vec{x}))$

- $f(y(\vec{x}), \vec{x}') = K(\vec{x}, \vec{x}') y(\vec{x}) \Rightarrow \frac{\delta F}{\delta y(\vec{x})} = K(\vec{x}, \vec{x}')$

- $f(y(\vec{x}), \vec{x}') = f(y(\vec{x})) \delta(\vec{x} - \vec{x}') \Rightarrow \text{OK}$

- $f(y(\vec{x}), \vec{\nabla} y(\vec{x})) = (\vec{\nabla} y(\vec{x}))^2 \Rightarrow - \sum_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial (\partial x_j)} \left(\sum_i \left(\frac{\partial y}{\partial x_i} \right)^2 \right)$
 $= -2 \sum_j \frac{\partial}{\partial x_j} \left(\frac{\partial y}{\partial x_j} \right)$
 $= -2 \vec{\nabla}^2 y(\vec{x})$

2. Proof of the chain rule:

- from the discrete point of view:

$$\frac{\partial f}{\partial h_i} = \sum_j \frac{\partial f}{\partial y_j} \cdot \frac{\partial y_j}{\partial h_i} \xrightarrow{i \mapsto \vec{x}, j \mapsto \vec{x}, y \mapsto y(\vec{x}), h \mapsto h(\vec{x})} \frac{\delta F}{\delta h(\vec{x})} = \int d\vec{x}' \frac{\delta F}{\delta y(\vec{x}')} \frac{\delta y(\vec{x}')}{\delta h(\vec{x})}$$

- from the definition of functionals:

we have $h \mapsto h + \delta h$ induces

$$y \mapsto y + \delta y \quad \text{with} \quad \delta y(\vec{x}') = \int d\vec{x} \frac{\delta y(\vec{x})}{\delta h(\vec{x})} \delta h(\vec{x})$$

$$F \mapsto F + \delta F \quad \text{with} \quad \delta F = \int d\vec{x}' \frac{\delta F(\vec{x}')}{\delta y(\vec{x}')} \delta y(\vec{x}')$$

$$= \int d\vec{x}' \frac{\delta F}{\delta y(\vec{x}')} \int d\vec{x} \frac{\delta y(\vec{x}')}{\delta h(\vec{x})} \delta h(\vec{x})$$

$$= \int d\vec{x} \underbrace{\left\{ \int d\vec{x}', \frac{\delta F}{\delta y(\vec{x}')} \frac{\delta y(\vec{x}')}{\delta h(\vec{x})} \right\}}_{\text{"}\frac{\delta F}{\delta h(\vec{x})}\text{"}} \delta h(\vec{x})$$

- application of chain rule

$$F[h] = \int d\vec{x}' e^{+\int d\vec{x} h(\vec{x}) w(\vec{x}, \vec{x}')}}$$

let us denote $y[h](\vec{x}') = \int d\vec{x} h(\vec{x}) w(\vec{x}, \vec{x}')$

$$\text{then } F[h] = \int d\vec{x}' e^{+y[h](\vec{x}')} \quad \text{with} \quad \frac{\delta y(h)(\vec{x}')}{\delta h(\vec{x})} = w(\vec{x}, \vec{x}')$$

$$\begin{aligned} \text{and } \frac{\delta F}{\delta h(\vec{x})} &= \int d\vec{x}' \left(+ e^{+y[h](\vec{x}')} \right) w(\vec{x}, \vec{x}') \\ &= + \int d\vec{x}' w(\vec{x}, \vec{x}') e^{+\int d\vec{x}'' h(\vec{x}'') w(\vec{x}'', \vec{x}')}} \end{aligned}$$

3. by guess:

$$F[y] = \int d\vec{x} \left\{ \frac{1}{2} (\vec{\nabla} V)^2 - \frac{P(\vec{x})}{\epsilon_0} V(\vec{x}) \right\}$$

$$\Rightarrow \frac{\delta F}{\delta V(\vec{x})} = -\frac{2}{2} \vec{\nabla}^2 V(\vec{x}) - \frac{P(\vec{x})}{\epsilon_0} = 0$$

Exponential of integral

let $F[y] = e^{-\int d\vec{x} y(\vec{x}) h(\vec{x})}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\vec{x}_1 \dots d\vec{x}_n y(\vec{x}_1) \dots y(\vec{x}_n) h(\vec{x}_1) \dots h(\vec{x}_n)$$

shows that

$$\frac{\delta^n F}{\delta y(\vec{x}_1) \dots \delta y(\vec{x}_n)} = (-1)^n h(\vec{x}_1) \dots h(\vec{x}_n)$$

n -variable function

On the existence and nature of solutions

let $F[y] = \int_{-1}^1 y^2(x)(1-y'(x))^2 dx$ with $y(-1)=0$ and $y(1)=1$

since y is differentiable, y is continuous, not necessarily y' .

1st method:

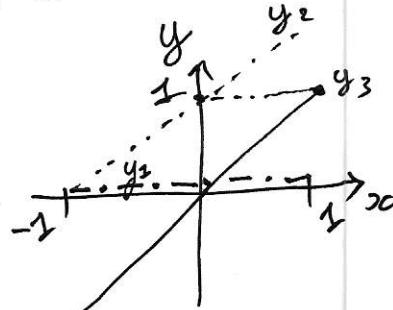
we observe that clearly $F[y] \geq 0 \quad \forall y$.

To realize the minimum, it is sufficient to cancel the integrand: $y(x)=0$ or $1-y'(x)=0$

the second yields $y(x)=x+a$ with a a constant with boundary conditions

$$y(-1)=0 \Rightarrow \begin{cases} y_1=0 \\ \text{or} \\ y_2=x+1 \end{cases}$$

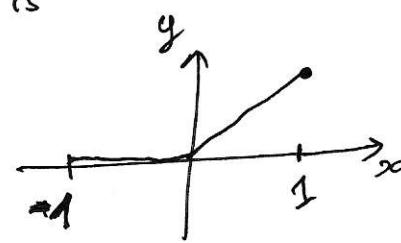
$$y(1)=0 \Rightarrow \begin{cases} y_3=x \end{cases}$$



y_2 and y_3 do not cross so they are incompatible

y_1 and y_3 cross at zero so the solution is

$$y(x) = \begin{cases} 0 \text{ for } x \in [-1, 0] \\ x \text{ for } x \in [0, 1] \end{cases}$$



then $y'(x)$ is discontinuous at $x=0$.

2nd method: $y_x = y'(x)$; $y_{xx} = y''(x)$

using Euler-Lagrange equation: $f(y, y_x) = y^2(1-y_x)^2$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_x} = 2y(1-y_x)^2 - \frac{d}{dx}(y^2(-2(1-y_x)))$$

$$= 2y(1-y_x^2) + \underbrace{2yy_x(1-y_x)}_{=0} - 2y^2y_{xx}$$

$$\rightarrow y(1-2y_x+y_x^2) + \cancel{2yy_x} - \cancel{y^2y_{xx}} = 1-2y_x+y_x^2+2y_x-2y_x^2$$

so we arrive at

$$y[-yy_{xx} + 1 - y_x^2] = 0$$

if $y \neq 0$: we have $yy_{xx} = 1 - y_x^2$ since $\frac{d}{dx}yy_x = y_x^2 + yy_{xx}$

it gives $\frac{d}{dx}(yy_x) = 1 = \frac{1}{2} \frac{d^2}{dx^2}(y^2)$

so $\frac{d}{dx}y^2(x) = 2x + a$ and $y^2(x) = \cancel{x^2} + ax + b$

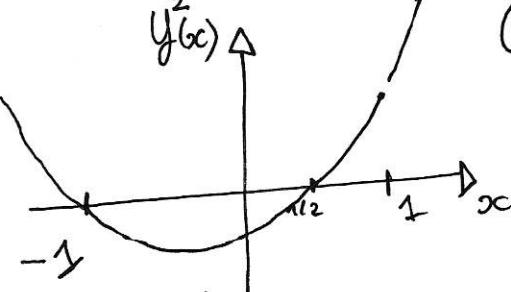
with boundary conditions: $y(-1) = 0 \Rightarrow \cancel{\frac{1}{2}} - a + b = 0$

$$y(1) = 1 \Rightarrow 1 = 1 + a + b \Rightarrow \left\{ \begin{array}{l} a = -b \\ = -1/2 \end{array} \right.$$

solution is

$$y^2 = x^2 + \frac{x}{2} - \frac{1}{2}$$

$$= (x+1)(x-1/2)$$



problem when $x \in [-1, 1/2] \cap y^2 < 0$

if we try $y=0$ for $x \in [-1, 1/2]$

and $y = \sqrt{(x+1)(x-1/2)}$ for $x \in [1/2, 1]$

we get

$$y_{xc} = \frac{1}{2} \frac{2x+1/2}{\sqrt{(x+1)(x-1/2)}}$$

$$\begin{aligned} \text{so } y^2(1-y_{xc})^2 &= (x+1)(x-1/2) \left(1 - \frac{x+1/4}{\sqrt{(x+1)(x-1/2)}}\right)^2 \\ &= \left(\sqrt{(x+1)(x-1/2)} - (x+1/4)\right)^2 > 0 \end{aligned}$$

so $F[y] > 0$ likely not to be the minimum.

let's take the problem again:

$y=0$ is possible

or $yy_{xc} = 1 - y_{xc}^2 \rightarrow$ another solution is $\begin{cases} y_{xc}=0 \\ y_{xc}^2=1 \end{cases}$

we are back to $y_{xc}=0$ solution and the ✓ result.

Change of boundary conditions:

we now consider $y(-1)=2$, $y(1)=1$, $y=0$ not possible

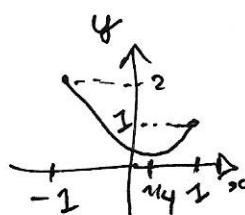
again, we try $y^2(x) = x^2 + ax + b$ $y_{xc}^2=1$ neither

which gives : $\begin{cases} 2 = 1 - a + b \\ 1 = 1 + a + b \end{cases} \Rightarrow a = -b = -1/2$

$$y^2(x) = x^2 - \frac{x}{2} + \frac{1}{2}$$

$$y_{xc} = \frac{1}{2} \frac{2(x-1/4)}{\sqrt{(x-1/4)^2 + 7/16}}$$

...



$$= \left(x - \frac{1}{4}\right)^2 - \frac{1}{16} + 1/2$$

$$= \left(x - \frac{1}{4}\right)^2 + 7/16 > 0$$

$F[y] > 0$ do not reach 0

Particular cases of Euler-Lagrange

1/ Beltrami identity:

(a) $F[y] = \int f(y, y_{xx}) dx$, Euler-Lagrange gives

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_{xx}} = 0$$

we have $\frac{d}{dx} \left(y_{xx} \frac{\partial f}{\partial y_{xx}} - f \right) = y_{xxx} \frac{\partial f}{\partial y_{xx}} + y_{xx} \frac{d}{dx} \frac{\partial f}{\partial y_{xx}} - \frac{\partial f}{\partial y} y_x - \frac{\partial f}{\partial y_{xx}} y_{xx}$

$$= 0$$

so $y_{xx} \frac{\partial f}{\partial y_{xx}} - f = \text{cste}$ (usually sets by particular points or boundary conditions)

(b) canonical momentum: $p = \frac{\partial L}{\partial \dot{q}}$; reminders on classical mechanics

action: $S[q] = \int_{t_i}^{t_f} dt L(q, \dot{q}, t)$; $L = T - V$ \rightarrow kinetic potential

principle of least action: $\frac{\delta S}{\delta q} = 0 \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$

p is the conjugate quantity of q w.r.t. L . From Euler-Lagrange, we get $\dot{p} = \frac{\partial L}{\partial \dot{q}}$ so \dot{p} is conjugate of q .

Then, intuitively we expect energy to be conserved in classical mechanics if $L(q, \dot{q}, t)$. We have from Beltrami that $p\dot{q} - L = \text{cste}$ (Legendre $\dot{q} \mapsto p$)

let us introduce the Hamiltonian $H = p\dot{q} - L$, then

$$dH = \cancel{pd\dot{q}} + \dot{q}dp - \underbrace{\frac{\partial L}{\partial q} dq}_{\dot{p}} - \underbrace{\frac{\partial L}{\partial \dot{q}} d\dot{q}}_{''p"} - \frac{\partial L}{\partial t} dt = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t} dt =$$

$\rightarrow H(q, p, t)$ (Legendre transform)

by identification:

$$\boxed{\dot{q} = \frac{\partial H}{\partial p}; \dot{p} = -\frac{\partial H}{\partial q}; \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

Hamilton's equation

"dissipation/absorption"

2/ Brachistochrone:

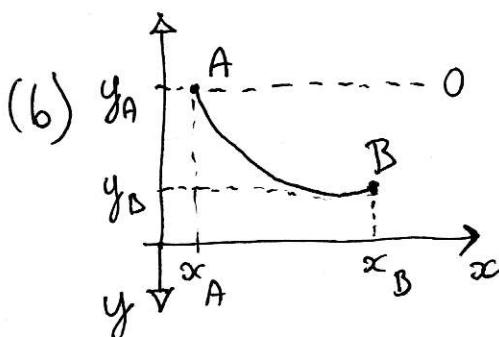
(a) from Euler-Lagrange:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_{xc}} = 0$$

||
0

$$\Rightarrow \boxed{\frac{\partial f}{\partial y_{xc}} = \text{const.}}$$

Not useful here



$$+ \dot{y}_A(t_A=0) = 0$$

time on the curve:

$$T = \int_{t_A=0}^T dt$$

along the curve:

$$\text{velocity: } v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v}$$

$$\text{with: } ds = \sqrt{dx^2 + dy^2} = dx \sqrt{1 + y_{xc}^2}$$

$$\begin{aligned} \text{energy conservation: } E &= \frac{1}{2}mv^2 + mgY \\ &= 0 \quad (\text{at point A}) \end{aligned}$$

$$\text{gives (assuming } v > 0\text{): } v = \sqrt{2gY}$$

So finally:

$$\boxed{T = \int_A^B \sqrt{\frac{1}{1+y_{xc}^2}} dx}$$

irrelevant

(c) Euler-equation would be heavy to write down.

we use Beltrami instead, but T is not the action and so the use do not correspond to energy conservation here.

with $f(y, y_{xc}) = \sqrt{\frac{1+y_{xc}^2}{y}}$, $y_{xc} \frac{2y_{xc}}{2\sqrt{y(1+y_{xc}^2)}} - \sqrt{\frac{1+y_{xc}^2}{y}} = \text{const} = K$

then $y_{xc}^2 - (1+y_{xc}^2) = K \sqrt{y(1+y_{xc}^2)}$

or $y(1+y_{xc}^2) = \text{const} = C$

(c) $y_{xc} = 1/\tan(\theta/2)$

gives $y = \frac{C}{1+\tan^2(\theta/2)} = C \sin^2(\frac{\theta}{2}) = \frac{C}{2}(1-\cos\theta)$

$$\Rightarrow y(\theta) = \frac{C}{2}(1-\cos\theta)$$

and $\frac{dx}{dy} = \tan(\theta/2)$, $dy = \frac{C}{2}\sin\theta d\theta$

$$\Rightarrow dx = \frac{C}{2}\tan(\theta/2)\sin\theta d\theta = 2\frac{C}{2}\sin^2(\frac{\theta}{2})d\theta$$

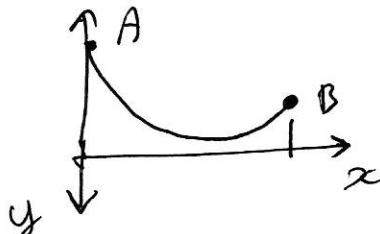
$$= \frac{C}{2}(1-\cos\theta)d\theta$$

$$\Rightarrow x(\theta) = \frac{C}{2}(\theta - \sin\theta)$$

+ one can show that

$$\theta(t) = \sqrt{\frac{2g}{C}}t$$

The curve is a cycloid



Shape of a soap film

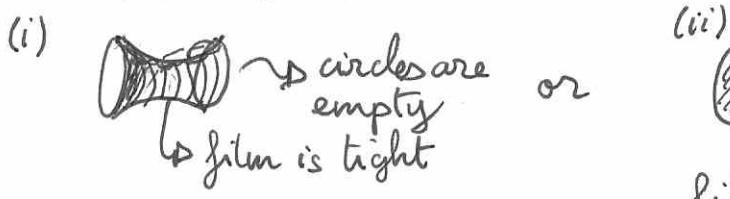
1.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1+y'^2} dx$$

$$dA = 2\pi y ds \rightarrow A = 2\pi \int_{x_1}^{x_2} y \sqrt{1+y'^2} dx$$

circle of perimeter $2\pi y$

2. we physically expect two situations:



case (i) is expected when films are close while (ii) is expected when we pull them apart, in case (ii),

$$A_0 = \pi(y_1^2 + y_2^2)$$
 is independent of x_1, x_2

3. Euler - Lagrange equation when $F = \int f(y, y') dx$

$$\frac{\partial f}{\partial y} - d \frac{\partial f}{\partial y'} = 0$$

$$\text{with here } f(y, y') = y \sqrt{1+y'^2}$$

gives $\sqrt{1+y'^2} - d \left(y \frac{y'}{\sqrt{1+y'^2}} \right) = \sqrt{1+y'^2} - \frac{y'^2 + yy''}{\sqrt{1+y'^2}} - \left(-\frac{yy'y'y''}{(1+y'^2)^{3/2}} \right)$
 $= 0$ multiplying by $(1+y'^2)^{3/2}$

$$\Rightarrow (1+y'^2)^2 - (1+y'^2)(y'^2 + yy'') + yy'^2y'' = 0$$

$$\text{or } 1 + 2y'^2 + y'^4 - y'^2 - yy'' - y'^4 - yy'^2y'' + yy'^2y'' = 0$$

$$\Rightarrow 1 + y'^2 = yy''$$

4. $y' = \sinh \theta \Rightarrow y'' = \theta' \cosh \theta \Rightarrow \frac{1 + \sinh^2 \theta}{\cosh^2 \theta} = y \theta' \cosh \theta$

so $y = \frac{\cosh \theta}{\theta'}$ taking the derivative of (2)
 $y' = -\frac{\theta''}{\theta'^2} \cosh \theta + \sinh \theta$
 $\stackrel{(1)}{=} \sinh \theta$

which gives $\theta'' = 0 \Rightarrow \theta(x) = ax + b$ with

a and b two integration constants. Since $\theta' = a$, injecting in (2) gives $y(x) = \frac{1}{a} \cosh(ax + b)$ with $a > 0$ since $y(0) > 0$.

a and b must be related to $(x_1, y_1), (x_2, y_2)$

5.
a. by symmetry, we must have $y(-x) = y(x) \Rightarrow b = 0$

b. both boundary conditions give $y(\pm d) = \frac{1}{a}$

$$d = \cosh(ad) \Leftrightarrow d(a) = \frac{\operatorname{argcosh}(a)}{a}$$

c. there exist a maximum for $d(a)$ located at (a^*, d^*) solution of the equation $d'(a) = 0$ ($\operatorname{argcosh}(a^*) = \frac{a^*}{\sqrt{a^{*2}-1}}$)

• if $d > d^*$: no solution for situation (i) \Rightarrow (ii) is realized

• if $d < d^*$: two solutions a_- and a_+ corresponding to shallow/deep curves

as $y''(0) = a$
shallow $\rightarrow y_- = \frac{1}{a_-} \cosh(a_- x)$ clearly y_- will have a smaller surface than y_+
deep $\rightarrow y_+ = \frac{1}{a_+} \cosh(a_+ x)$
d. A compared to A_0 .

Fluctuation - dissipation theorem in the path integral formalism

We have by definition:

$$Z[h] = \int \mathcal{D}\phi \ e^{-\beta(\mathcal{F}[\phi] - \int d\vec{x} \phi(\vec{x}) h(\vec{x}))}$$

↓ ↓
almost "free energy" coupling to the field
without field

$$F[h] = -\frac{1}{\beta} \ln Z[h]$$

computing averages and relation to derivative, as for discrete

$$Z(h_1, \dots, h_m) = \sum_{\ell} e^{-\beta(E_\ell - \vec{\phi}_\ell \cdot \vec{h})} \quad \sum_{j} \phi_{\ell,j} h_j \quad j = \text{spatial points}$$

$$\frac{\partial Z}{\partial h_j} = \sum_{\ell} \beta \phi_{\ell,j} e^{-\beta(E_\ell - \vec{\phi}_\ell \cdot \vec{h})}$$

$$\Rightarrow \langle \phi_j \rangle = \frac{1}{\beta Z} \frac{\partial Z}{\partial h_j}$$

in the continuum limit:

$$\frac{\delta Z}{\delta h(\vec{x})} = \beta \int \mathcal{D}\phi \ \phi(\vec{x}) e^{-\beta(\mathcal{F}[\phi] - \int d\vec{x}' \phi(\vec{x}') h(\vec{x}'))}$$

if not obvious \Rightarrow pedestrian way or, let us call

$$A[h] = e^{\beta \int d\vec{x} h(\vec{x}) \phi(\vec{x})} = f(I[h])$$

$$\text{with } f(t) = e^{\beta t} \text{ and } I[h] = \int d\vec{x} h(\vec{x}) \phi(\vec{x})$$

$$\text{then, } \frac{\delta A}{\delta h(\vec{x})} = f'(I[h]) \frac{\delta I}{\delta h(\vec{x})} = \phi(\vec{x}) \beta e^{\beta I[h]}$$

so

$$\frac{\delta Z}{\delta h(\vec{x})} = \beta \int \mathcal{D}\phi \ \phi(\vec{x}) e^{-\beta(\dots)}$$

□

Then, by definition of the average:

$$\langle \phi(\vec{x}) \rangle = \frac{1}{\beta Z} \frac{\delta Z}{\delta h(\vec{x})}$$

and

$$\langle \phi(\vec{x}) \rangle = -\frac{\delta F}{\delta h(\vec{x})}$$

Same idea: take a second time the derivative

$$\begin{aligned} \frac{\delta^2 Z}{\delta h(\vec{x}) \delta h(\vec{x}')} &= \beta \frac{\delta}{\delta h(\vec{x}')} \int D\phi \phi(\vec{x}) e^{-\beta(\mathcal{H}[\phi] - \int d\vec{x}'' h(\vec{x}'') \phi(\vec{x}''))} \\ &= \beta^2 \int D\phi \phi(\vec{x}) \phi(\vec{x}') e^{-\beta(\dots)} \\ &= \beta^2 Z \langle \phi(\vec{x}) \phi(\vec{x}') \rangle \end{aligned}$$

so that

$$\langle \phi(\vec{x}) \phi(\vec{x}') \rangle = \frac{1}{\beta^2 Z} \frac{\delta^2 Z}{\delta h(\vec{x}) \delta h(\vec{x}')}$$

Now,

$$\begin{aligned} \frac{\delta^2 F}{\delta h(\vec{x}) \delta h(\vec{x}')} &= -\frac{1}{\beta} \frac{\delta^2 \ln Z}{\delta h(\vec{x}) \delta h(\vec{x}')} = -\frac{1}{\beta} \frac{\delta}{\delta h(\vec{x})} \left(\frac{1}{Z} \frac{\delta Z}{\delta h(\vec{x}')} \right) \\ &= -\frac{1}{\beta} \left(-\frac{1}{Z^2} \left(\frac{\delta Z}{\delta h(\vec{x})} \right) \left(\frac{\delta Z}{\delta h(\vec{x}')} \right) + \frac{1}{Z} \frac{\delta^2 Z}{\delta h(\vec{x}) \delta h(\vec{x}')} \right) \\ &= -\beta \underbrace{\left(\langle \phi(\vec{x}) \phi(\vec{x}') \rangle - \langle \phi(\vec{x}) \rangle \langle \phi(\vec{x}') \rangle \right)}_{G(\vec{x}, \vec{x}')} = -\overline{\beta G(\vec{x}, \vec{x}')} \end{aligned}$$

$$\text{Last, } \chi(\vec{x}, \vec{x}') = \frac{\delta \langle \phi(\vec{x}) \rangle}{\delta h(\vec{x}')} = -\frac{\delta^2 F}{\delta h(\vec{x}) \delta h(\vec{x}')} = \beta G(\vec{x}, \vec{x}')$$

\Rightarrow

$$G(\vec{x}, \vec{x}') = \underbrace{\frac{1}{\beta} \chi(\vec{x}, \vec{x}')}_{\chi(\vec{x}, \vec{x}')}}$$