

Test on Complex analysis and Fourier transform

60mins

Thursday October 24th

You are allowed to use your notes and the summaries that I distributed.

Fourier transform of the hyperbolic secant

The hyperbolic secant function is defined as

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}} \quad (1)$$

and we here study some integrals related to its Fourier transform, in particular

$$I(\omega) = \int_0^\infty \frac{\cos(\omega x)}{\cosh(\pi x)} dx \quad \text{and} \quad J(\omega) = \int_0^\infty \frac{\sin(\omega x)}{\sinh(\pi x)} dx. \quad (2)$$

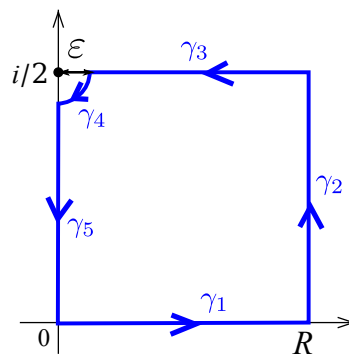


Figure 1: Sketch of the contour Γ split into elementary contours γ_j . The limits $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$ will be taken in the end.

We introduce $f(z) = \frac{e^{i\omega z}}{2 \cosh(\pi z)}$ and the contour Γ sketched in Fig. 1. We use the notation $I_j = \int_{\gamma_j} f(z) dz$ and $I_\Gamma = \oint_\Gamma f(z) dz$.

1. What are the poles of $f(z)$?
2. What is the value of I_Γ ?
3. Give the parametrization of the complex variable z for each subcontour γ_j .
4. Show that $I_2 \rightarrow 0$ when $R \rightarrow \infty$.

5. Give the expression of the real part of I_3 in the limit $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$ as a function of ω and $J(\omega)$.
6. Compute I_4 by an explicit calculation taking the $\varepsilon \rightarrow 0$ limit. Check that the same result could also be obtained using a well chosen contour and the residue theorem.
7. Show that the real part of I_5 is zero.
8. Conclude that one has the relation, for all real ω

$$I(\omega) - e^{-\omega/2} J(\omega) = \frac{e^{-\omega/2}}{2} \quad (3)$$

9. By considering the relation for $-\omega$, infer the explicit expression of $I(\omega)$ and $J(\omega)$.
10. Compute the following Fourier transform, with $a > 0$ a real number

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \text{sech}(ax) e^{-i\omega x} dx \quad (4)$$

11. What is the peculiarity of the hyperbolic secant with respect to Fourier transform? Do you know other examples of functions sharing this property?