Fourier transform of the hyperbolic secant

1. There are only simple poles along the imaginary axis $z_n = (1/2+n)i$ with $n \in \mathbb{Z}$ since $\cosh(it) = \cos(t)$.

2. $f(z)$ is holomorphic on and inside Γ that is simply connected so from Cauchy theorem $I_\Gamma = 0$.

3. γ_1: $z = x, \quad x \in [0, R]$
   γ_2: $z = R + it, \quad t \in [0, 1/2]$
   γ_3: $z = x + i/2, \quad x \in [R, \varepsilon]$
   γ_4: $z = i/2 + \varepsilon e^{i\theta}, \theta \in [0, -\pi/2]$
   γ_5: $z = it, \quad t \in [1/2 - \varepsilon, 0]$

4. We have $2|\cosh(\pi(R+it))| = \left|e^{\pi R + i\pi t} + e^{-\pi R - i\pi t}\right| \geq \left|\left|e^{\pi R+i\pi t}\right| - \left|e^{-\pi R - i\pi t}\right|\right| = 2\sinh(\pi R)$ and $\left|e^{i\omega(R+it)}\right| = e^{-\omega t}$ so $|f(z)| \leq e^{-\omega t}/(2\sinh(\pi R))$ which kills the integral on this finite domain. So $I_2 \to 0$ when $R \to \infty$.

5. With $\cosh(\pi(x+i/2)) = i\sinh(\pi x)$, one obtains $\mathrm{Re}(I_3) = -\dfrac{e^{-\omega/2}}{2}\displaystyle\int_\varepsilon^R \frac{\sin(\omega x)}{\sinh(\pi x)}dx \to -\frac{e^{-\omega/2}}{2}J(\omega)$. Notice that the imaginary has a logarithmic divergence when $\varepsilon \to 0$.

6. Using the parametrization, we explicitly have when $\varepsilon \to 0$
$$I_4 = \int_0^{-\pi/2} d\theta\, i\varepsilon e^{i\theta} \frac{e^{-\omega/2 + i\varepsilon e^{i\theta}}}{2\cosh(\pi i/2 + \pi\varepsilon e^{i\theta})} = e^{-\omega/2}\int_0^{-\pi/2} d\theta \frac{i\varepsilon e^{i\theta}}{2i\sinh(\pi\varepsilon e^{i\theta})} = -\frac{e^{-\omega/2}}{4} \tag{1}$$
   By choosing a circle of radius ε with anti-trigonometric direction, the contour encircles the simple pole $z_0 = i/2$ so by applying the residue theorem, one has
$$\oint_{\circlearrowright} f(z)dz = -2i\pi \mathrm{Res}(f, i/2) = -2i\pi \frac{e^{-\omega/2}}{2\pi\sinh(\pi i/2)} = -e^{-\omega/2} \tag{2}$$
   and then, assuming that $\varepsilon \to 0$ to make the integrand angle-independent, one has $I_4 = 1/4 \oint_{\circlearrowright} f(z)dz$

7. One gets that $I_5 = -i\displaystyle\int_0^{1/2-\varepsilon} \frac{e^{-\omega t}}{2\cosh(\pi t)}dt$ is a pure imaginary number.

8. From $\mathrm{Re}(I_\Gamma) = 0$, collecting the terms with $\mathrm{Re}\{I_1\} = I(\omega)/2$ yields $I(\omega) - e^{-\omega/2}J(\omega) = \dfrac{e^{-\omega/2}}{2}$.

9. Since $I(-\omega) = I(\omega)$ and $J(-\omega) = -J(\omega)$, we have $I(\omega) + e^{\omega/2}J(\omega) = \dfrac{e^{\omega/2}}{2}$. Combining the two equations yields
$$I(\omega) = \frac{1}{2\cosh(\omega/2)} \quad \text{and} \quad J(\omega) = \frac{1}{2}\tanh(\omega/2)\ . \tag{3}$$

10. One gets, after seeing that $\int_{-\infty}^{+\infty} \frac{\sin(\omega x)}{\cosh(a x)}dx = 0$ and setting $x' = (a/\pi)x$
$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\cos(\omega x)}{\cosh(ax)}dx = \frac{1}{\sqrt{2\pi}}\frac{\pi}{a}2I\left(\frac{\pi}{a}\omega\right) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\,\mathrm{sech}\left(\frac{\pi}{2a}\omega\right) \tag{4}$$

11. As the gaussian (and several other functions), the hyperbolic secant is its own Fourier transform up to some rescaling factor.