Test on Calculus of variations

60mins

Thursday October 3rd

Field theories

Elastic string – We consider a problem described by a **scalar** field $\psi(x,t)$ which equations of motion are obtained by the minimization of the action $\mathcal{S}[\psi]$, expressed in terms of the Lagrangian density $\mathcal{L}$ as

$$\mathcal{S}[\psi] = \int_0^L \int_0^T dx\, dt\, \mathcal{L}(\psi, \partial_t\psi, \partial_x\psi) \ , \tag{1}$$

with L and T some fixed parameters.

1. Write down the Euler-Lagrange equation $\frac{\delta S}{\delta\psi(x,t)} = 0$ in terms of partial derivatives of $\mathcal{L}$.

2. We consider the Lagrangian density $\mathcal{L} = \dfrac{\rho}{2}\left((\partial_t\psi)^2 - c^2(\partial_x\psi)^2\right)$ with ρ and c two constants. What is the equation of motion for $\psi(x,t)$? What is the physical meaning of c?

Non-linear Schrödinger equation – We now consider a **complex** field $\psi(\vec{x},t)$ describing a quantum problem with $\psi^*(\vec{x},t)$ its complex conjugate. The position variable $\vec{x}$ is now in arbitrary dimension. One can take either the real and imaginary part of ψ as independent variables or, equivalently, use ψ and ψ^* as independent variables. Thus, the action and Lagrangian density takes the form

$$\mathcal{S}[\psi,\psi^*] = \int \int_0^T d\vec{x}\, dt\, \mathcal{L}(\psi, \psi^*, \partial_t\psi, \partial_t\psi^*, \vec{\nabla}\psi, \vec{\nabla}\psi^*) \ , \tag{2}$$

with $(\vec{\nabla})_j = \partial_{x_j}$.

3. Write down the Euler-Lagrange equations for the field $\psi(\vec{x},t)$.

4. We consider the following Lagrangian density with $V(x)$ a potential and g a constant:

$$\mathcal{L} = i\frac{\hbar}{2}\left(\psi^*\,\partial_t\psi - \psi\,\partial_t\psi^*\right) - \frac{\hbar^2}{2m}\vec{\nabla}\psi\cdot\vec{\nabla}\psi^* - V(x)|\psi|^2 - g|\psi|^4. \tag{3}$$

   What are the equations of motion for $\psi(\vec{x},t)$? How are they related?

Optimal ski trajectory

We consider a skier going from point O to A (see Fig. 1) on a slope that makes an angle α with the horizontal plane. The skier is submitted only to the gravity field that corresponds to a potential energy $-xmg\sin\alpha$ with m its mass, g the gravity constant and O taken to be the origin of potential energy. We want to find the trajectory in the xy plane that minimizes the total time T to reach A such that the skier starts with zero velocity.

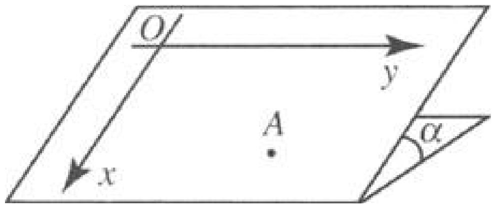

Figure 1: Geometry and notations for the ski slope.

1. Write down the total energy as a function of variables x, y and the velocity $\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$. What is the value of the total energy?

2. We introduce $y'(x) = \frac{dy}{dx}$. Show that the total time reads

$$T = \frac{1}{\sqrt{2g\sin\alpha}} \int_0^{x_A} dx \sqrt{\frac{1+y'^2}{x}} \tag{4}$$

3. Determine the differential equation of motion satisfied by $y(x)$.

4. Show that the quantity $\dfrac{1}{x}\dfrac{dy}{dt}$ is independent of time.

5. Check that $x(\theta) = (1-\cos 2\theta)/(2C^2)$ and $y(\theta) = (2\theta - \sin(2\theta))/(2C^2)$, with $\theta(t)$ a function of time, is a parametric solution for the curve and give the value of the constant C .

6. Find the $\theta(t)$ function.