Field theories

Elastic string –

1. $$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial_x \psi)} = 0$$

2. $\rho\left(-\frac{\partial}{\partial t}\partial_t\psi + c^2\frac{\partial}{\partial x}\partial_x\psi\right) = 0$ so $\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = 0$, d'Alembert equation with velocity c.

Non-linear Schrödinger equation –

$$\mathcal{S}[\psi,\psi^*] = \int_0^L\int_0^T dx\,dt\,\mathcal{L}(\psi,\psi^*,\partial_t\psi,\partial_t\psi^*,\vec{\nabla}\psi,\vec{\nabla}\psi^*)\ , \tag{1}$$

with $(\vec{\nabla})_j = \partial_{x_j}$.

3. There are two Euler-Lagrange equations $\frac{\delta\mathcal{S}}{\delta\psi} = 0$ and $\frac{\delta\mathcal{S}}{\delta\psi^*} = 0$. One has, following the lecture,

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} - \sum_j \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial(\partial_{x_j}\psi)} = 0 \quad\text{and}\quad \frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \psi^*)} - \sum_j \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial(\partial_{x_j}\psi^*)} = 0 \tag{2}$$

4. we get, using $\vec{\nabla}\psi\cdot\vec{\nabla}\psi^* = \sum_j(\partial_{x_j}\psi)(\partial_{x_j}\psi^*)$ and $|\psi|^2 = \psi\,\psi^*$

$$i\frac{\hbar}{2}\left(-\partial_t\psi^* - \frac{\partial}{\partial t}\psi^*\right) + \frac{\hbar^2}{2m}\sum_j\frac{\partial}{\partial x_j}(\partial_{x_j}\psi^*) - V(x)\psi^* - 2g\psi(\psi^*)^2 = 0 \tag{3}$$

$$i\frac{\hbar}{2}\left(\partial_t\psi + \frac{\partial}{\partial t}\psi\right) + \frac{\hbar^2}{2m}\sum_j\frac{\partial}{\partial x_j}(\partial_{x_j}\psi) - V(x)\psi - 2g\psi^*\psi^2 = 0 \tag{4}$$

Both equations are actually related by complex conjugation and gives the non-linear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi + 2g|\psi|^2\psi \tag{5}$$

Optimal ski trajectory

1. $E = \frac{1}{2}m\vec{v}^{\,2} - mg\sin\alpha\,x = 0$ using the initial conditions at the point O.

2. We have $v = |\vec{v}| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \frac{dx}{dt}\sqrt{1+y'^2} = \sqrt{2g\sin\alpha\,x}$ since we take $dx > 0$ and $dy > 0$ to reach A. Then, the total time is a functional of $y(x)$ with

$$T[y] = \int_0^T dt = \frac{1}{\sqrt{2g\sin\alpha}}\int_0^{x_A} dx\sqrt{\frac{1+y'^{\,2}}{x}} \tag{6}$$

3. Using Euler Lagrange equation, since there is no explicit dependence on y for $f(y,y',x) = \sqrt{\frac{1+y'^2}{x}}$, we get

$$-\frac{\mathrm{d}}{\mathrm{d}x}\frac{y'}{\sqrt{x(1+y'^{\,2})}} = 0 \tag{7}$$

A first integration leads to $y' = B\sqrt{x(1+y'^{\,2})}$ with $B > 0$ some constant. Then, we get $y'^{\,2} = \dfrac{B^2x}{1-B^2x}$.

4. We have $\frac{1}{x}\frac{dy}{dt} = \frac{y'}{x}\frac{dx}{dt} = \frac{y'}{x}\sqrt{\frac{2g\sin\alpha\,x}{1+y'^{\,2}}} = B\sqrt{2g\sin\alpha}$.

5. We have $\frac{\mathrm{d}x}{\mathrm{d}t} = \sin 2\theta / C^2 \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = (1 - \cos(2\theta))/C^2 \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ and $x = \sin^2\theta / C^2$. Thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-\cos(2\theta)}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \tan\theta$ and $\dfrac{B^2 x}{1 - B^2 x} = \dfrac{B^2 \sin^2\theta}{C^2 - B^2\sin^2\theta} = \tan^2\theta$ provided $C = B$. This shows that this indeed is the solution.
6. We clearly see that $\dfrac{1}{x}\dfrac{dy}{dt} = 2\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = B\sqrt{2g\sin\alpha}$ so that $\theta(t) = B\sqrt{2g\sin\alpha}\, t/2$. The curve is a branch of cycloid. The value of B is determined by the coordinates of the point A (not asked).