Test on Calculus of variations and Complex analysis

60mins

Thursday October 21th

Sturm-Liouville variational problem

We consider the functional

$$F[y] = \int_a^b \mathrm{d}x \, \frac{1}{2} \left(p(x) y'(x)^2 - m(x) y(x)^2 \right) \tag{1}$$

on a space of analytic functions y over $[a, b]$ and satisfying to the boundary conditions $y(a) = y(b) = 0$. The functions $p(x)$ and $m(x)$ are simple polynomials. We look for the extremalization of F under the constraint that y is normalized, ie such that

$$\int_a^b y^2(x) \mathrm{d}x = 1 \tag{2}$$

1. Show that the solutions y statisfy to the differential eigenvalue problem

$$\hat{L} y = -\lambda y \tag{3}$$

in which $\hat{L}$ is a second order differential operator and λ an eigenvalue. In particular, gives the expression of $\hat{L}$ in terms of operators $\frac{\mathrm{d}}{\mathrm{d}x}$ and functions $p(x)$ and $m(x)$.

2. We take $p(x) = 1 - x^2$ and $m(x) = 0$. Give the explicit form for $\hat{L}$.

A Fourier transform

Compute the following principal part of the Fourier transform

$$F(k) = \mathrm{PP} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x(1 + x^2)} \mathrm{d}x \tag{4}$$

Some complex gaussian integrals

We wish to compute the following integral

$$I(a,b) = \iint \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2i\pi}\, e^{-a\,z\,\bar{z}-b\bar{z}-\bar{b}z} \tag{5}$$

where $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$ and $b \in \mathbb{C}$. The notation $\mathrm{d}z\mathrm{d}\bar{z}$ means integrating overs two infinite lines parametrized by $x, y \in \mathbb{R}$ such that $z = x + iy$ so that we have $\mathrm{d}z\mathrm{d}\bar{z} = 2i\mathrm{d}x\mathrm{d}y$.

1. Rewrite $I(a,b)$ in terms of a double integral over x and y.
2. We see that we need to generalize the results on gaussian integrals. We will show that

$$I(a) = \int_{-\infty}^{\infty} e^{-a(x+c)^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}} \quad \text{with} \quad c = c^r + ic^i \in \mathbb{C}. \tag{6}$$

   a) First show the result for $c = 0$. To do so, either consider the trick of writting $I^2(a)$ in polar coordinates or use a well chosen contour in the complex plane.

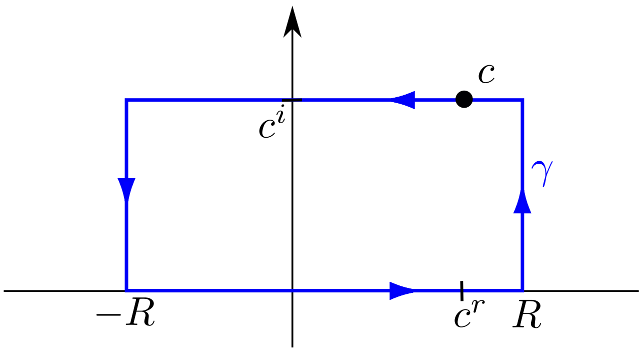

Figure 1: The contour γ to compute (6).

   b) For $c \neq 0$, use the contour of Fig. 1 to prove (6).

   c) Application to Fresnel integrals : infer the values of the integrals

$$F_c = \int_{-\infty}^{\infty} \cos\!\left(x^2\right)\mathrm{d}x \quad \text{and} \quad F_s = \int_{-\infty}^{\infty} \sin\!\left(x^2\right)\mathrm{d}x \tag{7}$$

3. Finally compute the expression of $I(a,b)$ in (5) as a function of a and b.

Derivative of the principal part

We introduce the finite part FP defined as

$$\mathrm{FP}\int_{-\infty}^{+\infty} \frac{f(x)-f(x_0)}{(x-x_0)^2}\mathrm{d}x = \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{x_0-\epsilon}\frac{f(x)}{(x-x_0)^2}\mathrm{d}x + \int_{x_0+\epsilon}^{+\infty}\frac{f(x)}{(x-x_0)^2}\mathrm{d}x - \frac{2f(x_0)}{\epsilon}\right) \tag{8}$$

What is the derivative $\dfrac{\mathrm{d}}{\mathrm{d}x_0}\mathrm{PP}\displaystyle\int_{-\infty}^{+\infty}\frac{f(x)}{x-x_0}\mathrm{d}x$? Provide another expression involving $f'(x)$ and a principal part.